2.4 Orthogonal Coordinate Systems

Reading Assignment: *pp.16-33*

We live in a **3**-dimensional world!

Meaning:

1)

2)

Q: What 3 scalar **values** and what 3 unit **vectors** do we use ??

A: We have several options! A **set** of 3 scalar values that define position and a set of unit vectors that define direction form a **Coordinate system**. Examples of coordinate systems include:

1.

2.

3.

A. Coordinates

* The 3 **scalar** values used to define **position** are called **coordinates**.

* E.G., scalar values u_1, u_2, and u_3 can define precisely the **location** of point P in space (i.e., P(u_1, u_2, u_3)).

* All coordinates are defined with respect to an **arbitrary** point called the **origin**.

HO: Cartesian Coordinates

HO: Cylindrical Coordinates

HO: Spherical Coordinates

B. Coordinate Transformations

We can rewrite the **location** of point P(x,y,z) in terms of cylindrical coordinates (i.e, P(r,θ,ϕ)), for example.

Or, we can rewrite a **scalar field** $g(x,y,z)$ in terms of cylindrical coordinates (i.e, $g(\rho,\phi,z)$), for example.

HO: Coordinate Transformations

Example: Coordinate Transformations

C. Base Vectors

* The 3 **unit vectors** used to define **direction** are called **base vectors**.

* E.G., base vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ can be used to precisely describe the **direction** of some **vector A**.

HO: Base Vectors

HO: Cartesian Base Vectors

D. Vector Expansion using Base Vectors

Q: **Why** are base vectors important? **How** are they used?

A: We find that any and **all** vectors can be expressed as the **sum** of **3** vectors, each pointing in the precise **direction** of one of the three base vectors!

e.g.,

$$\mathbf{B} = B_1\,\hat{\mathbf{a}}_1 + B_2\,\hat{\mathbf{a}}_2 + B_3\,\hat{\mathbf{a}}_3$$

or

$$\mathbf{C} = C_x\,\hat{\mathbf{a}}_x + C_y\,\hat{\mathbf{a}}_y + C_z\,\hat{\mathbf{a}}_z$$

HO: Vector Expansion using Base Vectors

E. Spherical and Cylindrical Base Vectors

HO: Spherical Base Vectors

HO: Cylindrical Base Vectors

F. Vector Algebra and Vector Expansions

HO: Vector Algebra using Orthonormal Base Vectors

G. The Vector Field

* Recall a vector **field** is a function of **position**.

* We express position in terms of **coordinates**.

* Thus, a vector field is **function** of coordinate values (e.g., *x*, *y*, *z*).

* But, we express a vector field with **3 scalar components**.

HO: Vector Fields

HO: Expressing Vector Fields with Coordinate Systems

H. The Position Vector

In addition to coordinates (e.g., r,θ,ϕ), we can use a special **directed distance** to specify points in space.

HO: The Position Vector

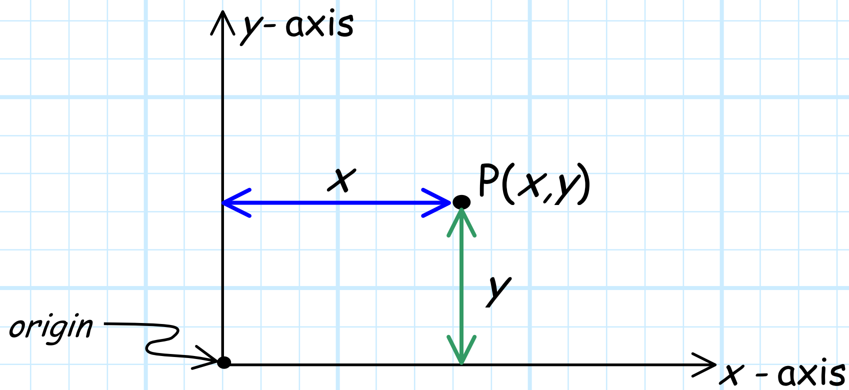
HO: Applications of the Position Vector

HO: Vector Field Notation

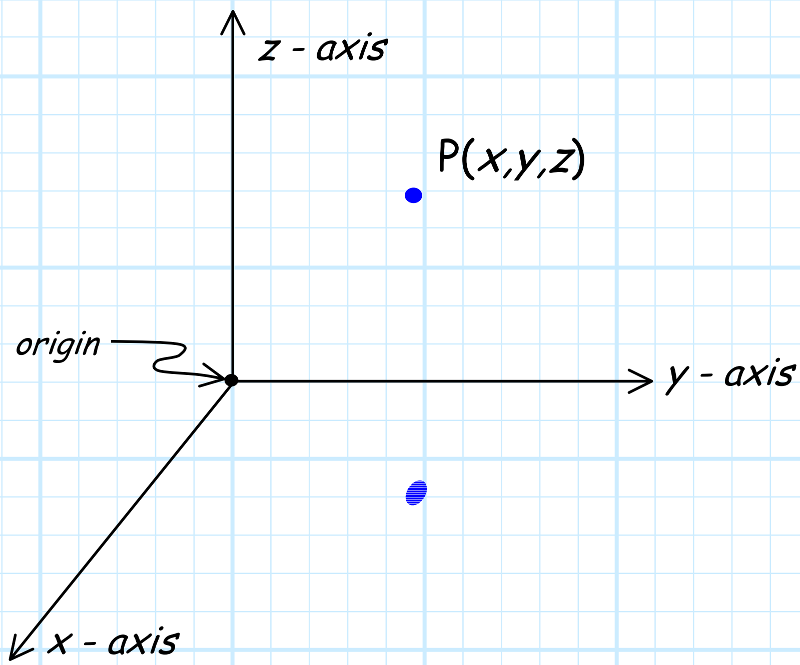
HO: A Gallery of Vector Fields

Cartesian Coordinates

You're probably familiar with **Cartesian coordinates**. In **two**-dimensions, we can specify a point on a plane using **two** scalar values, generally called *x* and *y*.



We can extend this to **three**-dimensions, by adding a **third** scalar value *z*.

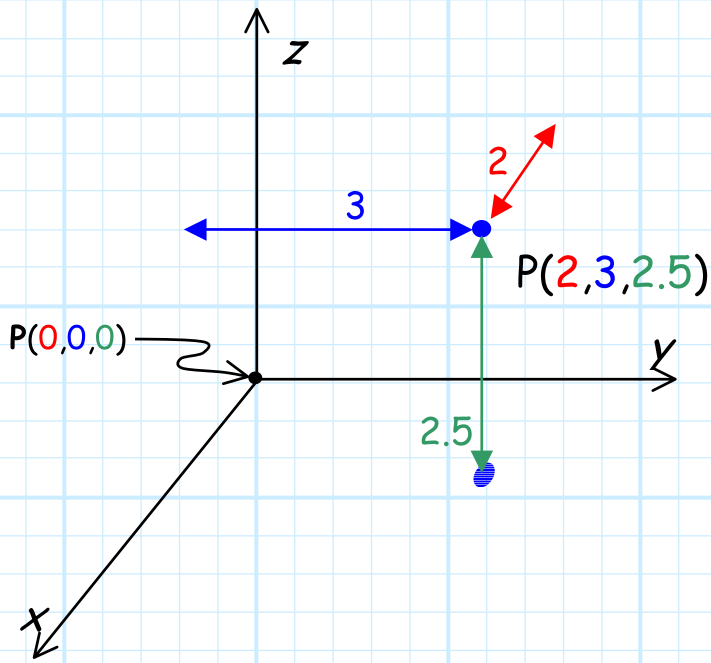


Note the coordinate values in the Cartesian system effectively represent the **distance** from a **plane** intersecting the origin.

For **example**, $x = 3$ means that the point is **3 units** from the *y-z* **plane** (i.e., the $x = 0$ plane).

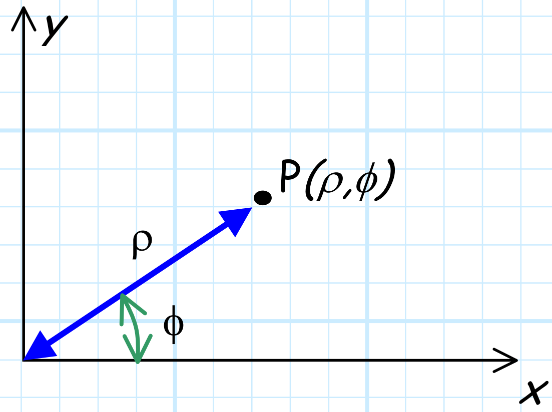
Likewise, the *y* coordinate provides the **distance** from the *x-z* (y=0) plane, and the *z* coordinate provides the **distance** from the *x-y* (z =0) plane.

Once **all three** distances are specified, the **position** of a point is **uniquely** identified.

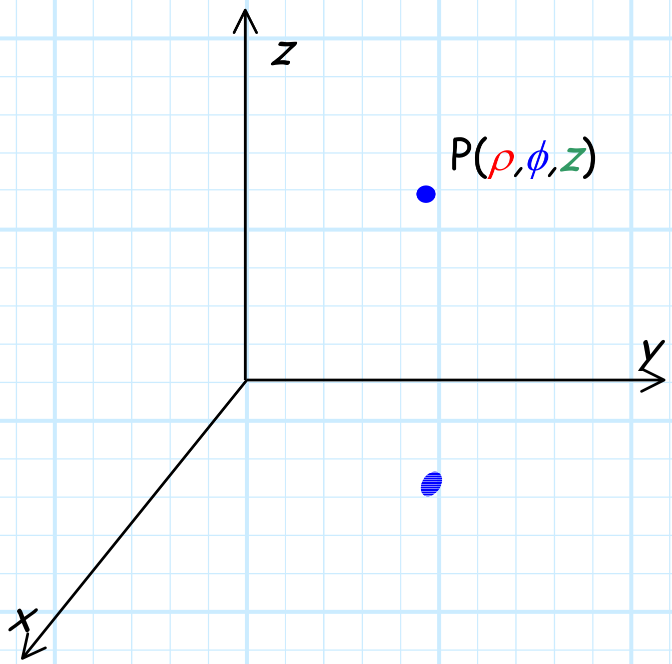


Cylindrical Coordinates

You're probably also familiar with **polar coordinates**. In **two**-dimensions, we can also specify a point with **two** scalar values, generally called ρ and ϕ.



We can extend this to **three**-dimensions, by adding a **third** scalar value *z*. This method for identifying the position of a point is referred to as **cylindrical coordinates**.



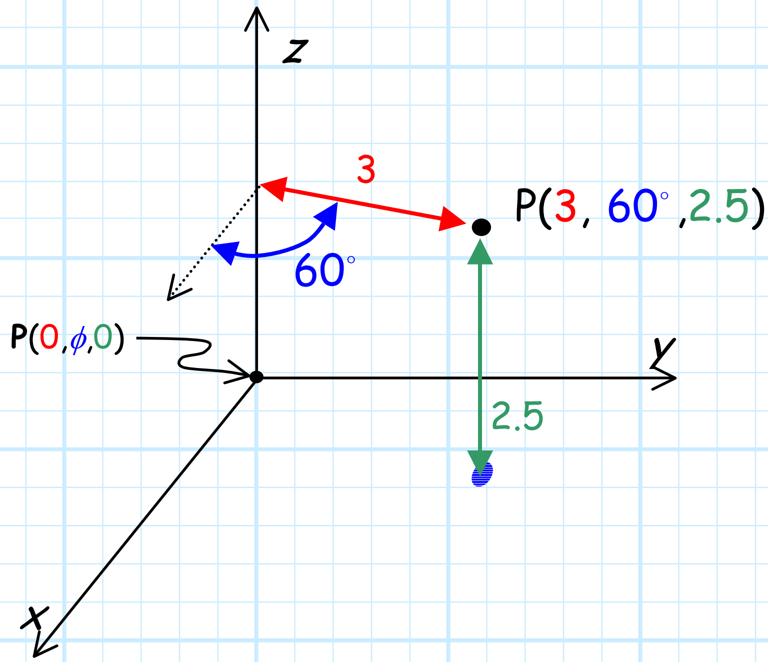
Note the **physical** significance of each parameter of **cylindrical** coordinates:

1. The value ρ indicates the **distance** of the point from the ***z*-axis** $(0 \leq \rho < \infty)$.

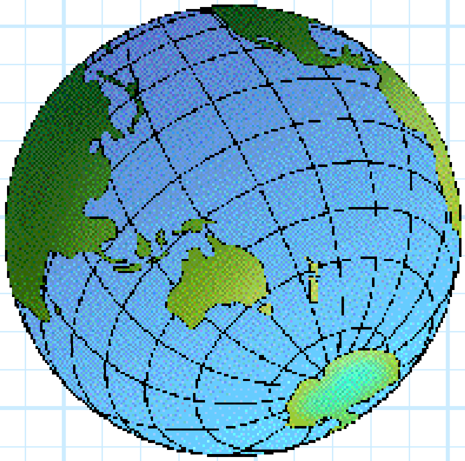
2. The value ϕ indicates the **rotation angle** around the ***z*-axis** $(0 \leq \phi < 2\pi)$, **precisely** the same as the angle ϕ used in **spherical** coordinates.

3. The value z indicates the **distance** of the point from the *x*-*y* (*z* = 0) plane $(-\infty < z < \infty)$, **precisely** the same as the coordinate z used in **Cartesian** coordinates

Once **all three** values are specified, the **position** of a point is **uniquely** identified.



Spherical Coordinates

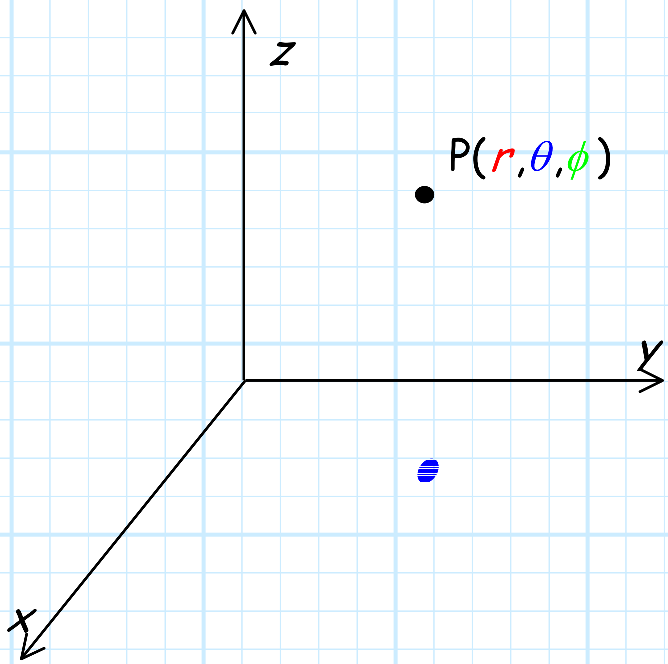


* **Geographers** specify a location on the Earth's surface using **three** scalar values: **longitude**, **latitude**, and **altitude**.

* Both longitude and latitude are **angular** measures, while altitude is a measure of **distance**.

* Latitude, longitude, and altitude are similar to **spherical coordinates**.

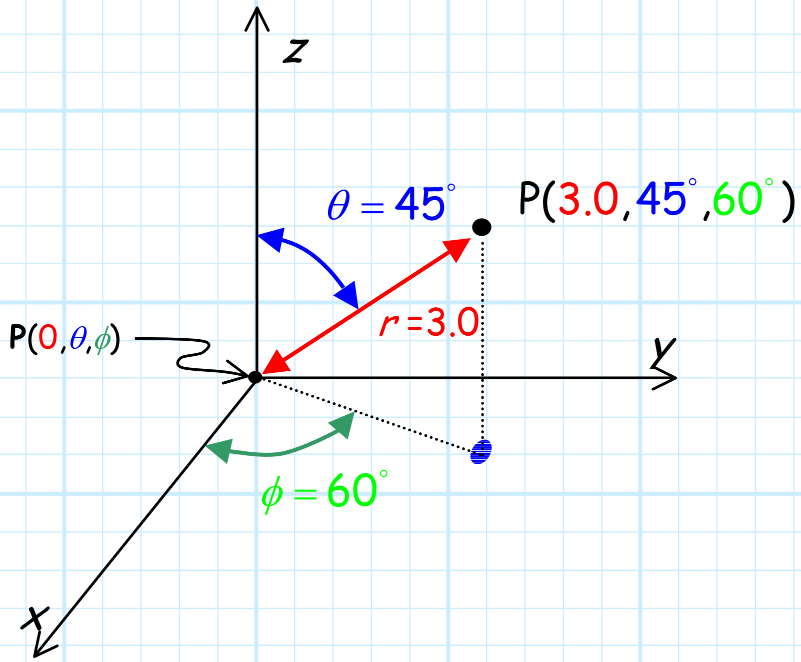
* Spherical coordinates consist of one scalar value (r), with units of **distance**, while the other two scalar values (θ,ϕ) have **angular** units (degrees or radians).



1. For spherical coordinates, r $(0 \le r < \infty)$ expresses the **distance** of the point from the **origin** (i.e., similar to **altitude**).

2. Angle θ $(0 \le \theta \le \pi)$ represents the angle formed **with the *z*-axis** (i.e., similar to **latitude**).

3. Angle ϕ $(0 \le \phi < 2\pi)$ represents the rotation angle around the *z*-axis, **precisely** the same as the **cylindrical** coordinate ϕ (i.e., similar to **longitude**).



Thus, using **spherical** coordinates, a point in space can be unambiguously defined by **one distance** and **two angles**.

Coordinate Transformations

Say we **know** the location of a point, or the description of some scalar field in terms of **Cartesian** coordinates (e.g., $T(x,y,z)$).

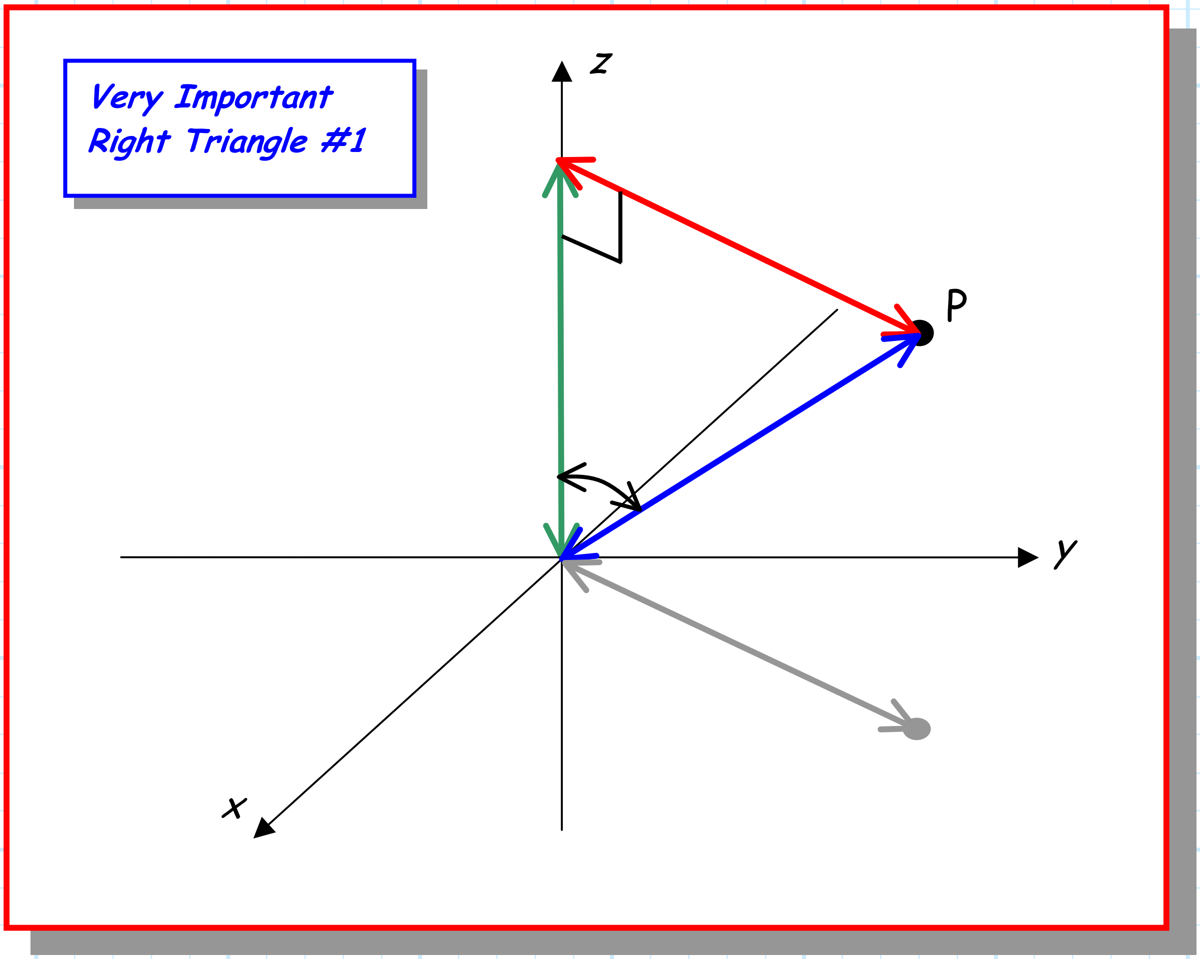
What if we decide to express this point or this scalar field in terms of **cylindrical** or **spherical** coordinates **instead**?

Q: *How do we accomplish this **coordinate transformation**?*

A: Easy! We simply apply our knowledge of **trigonometry**.

We see that the coordinate values z, ρ, r, and θ are all variables of a **right triangle**! We can use our knowledge of trigonometry to relate them to each other.

In fact, we can **completely derive** the relationship between **all six** independent coordinate values by considering just **two very important right triangles**! → **Hint:** ***Memorize these 2 triangles!!!***



It is evident from the triangle that, for example:

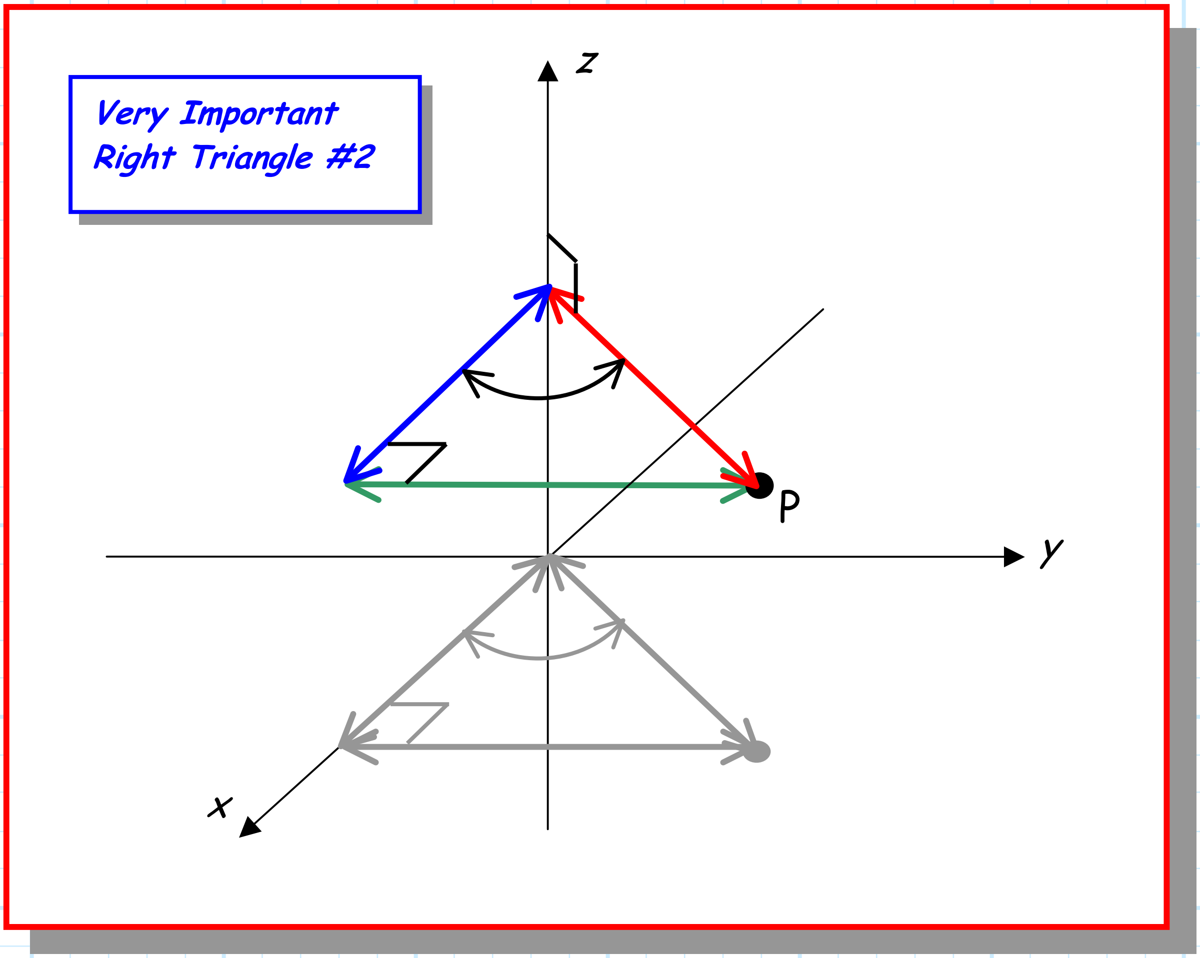
$$z = r\cos\theta = \rho\cot\theta = \sqrt{r^2 - \rho^2}$$

$$\rho = r\sin\theta = z\tan\theta = \sqrt{r^2 - z^2}$$

$$r = \sqrt{\rho^2 + z^2} = \rho\csc\theta = z\sec\theta$$

$$\theta = \tan^{-1}\left[\rho/z\right] = \sin^{-1}\left[\rho/r\right] = \cos^{-1}\left[z/r\right]$$

Likewise, the coordinate values x, y, ρ, and ϕ are **also** related by a **right triangle**!



From the resulting triangle, it is evident that:

$$x = \rho \cos\phi = y \cot\phi = \sqrt{\rho^2 - y^2}$$

$$y = \rho \sin\phi = x \tan\phi = \sqrt{\rho^2 - x^2}$$

$$\rho = \sqrt{x^2 + y^2} = x \sec\phi = y \csc\phi$$

$$\phi = \tan^{-1}\left[{}^{y}\!/_{x}\right] = \cos^{-1}\left[{}^{x}\!/_{\rho}\right] = \sin^{-1}\left[{}^{y}\!/_{\rho}\right]$$

Combining the results of the two triangles allows us to write each coordinate set in terms of each other:

Cartesian and Cylindrical

$$x = \rho \cos\phi \qquad \rho = \sqrt{x^2 + y^2}$$

$$y = \rho \sin\phi \qquad \phi = \tan^{-1}\left[\frac{y}{x}\right] \quad \text{(be careful !)}$$

$$z = z \qquad z = z$$

Cartesian and Spherical

$$x = r \sin\theta \cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r \sin\theta \sin\phi \qquad \theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right]$$

$$z = r \cos\theta \qquad \phi = \tan^{-1}\left[\frac{y}{x}\right]$$

Cylindrical and Spherical

$$\rho = r \sin\theta$$
$$\phi = \phi$$
$$z = r \cos\theta$$

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right]$$
$$\phi = \phi$$

Example: Coordinate Transformations

Say we have denoted a **point** in space (using **Cartesian** Coordinates) as $P(x=-3,\ y=-3,\ z=2)$.

Let's **instead** define this **same** point using **cylindrical** coordinates ρ, ϕ, z:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{-3}{-3}\right] = \tan^{-1}[1] = 45^\circ$$

$$z = 2$$

Therefore, the location of this point can **perhaps** be defined **also** as $P(\rho = 3\sqrt{2},\ \phi = 45^\circ,\ z = 2)$.

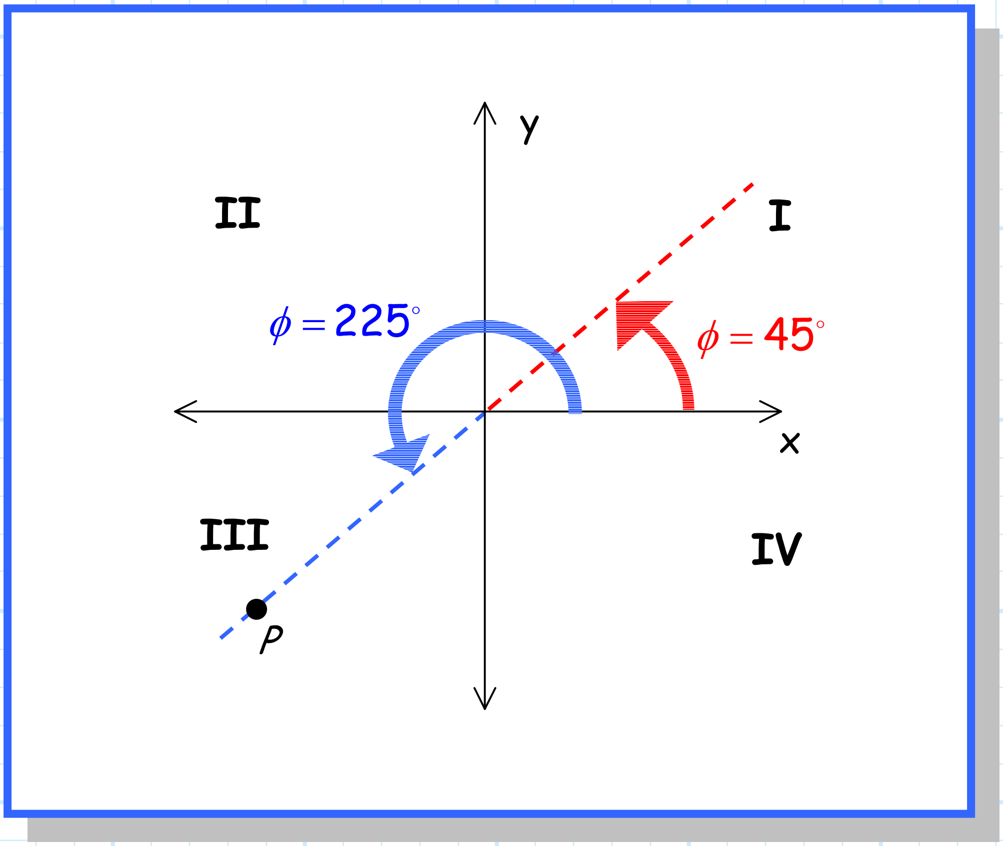
Q: ***Wait!*** *Something has gone **horribly wrong**. Coordinate $\phi = 45^\circ$ indicates that point P is located in **quadrant I**, whereas the coordinates x =-3, y =-3 tell us it is in fact in **quadrant III**!*

A: The problem is our interpretation of the **inverse tangent!**

Remember that $0 \leq \phi < 360^{\circ}$, so that we must do a **four quadrant** inverse tangent. Your calculator likely only does a **two quadrant** inverse tangent (i.e., $90 \leq \phi \leq -90^{\circ}$), so **be careful**!

Therefore, if we **correctly** find the coordinate ϕ:

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{-3}{-3}\right] = 225^{\circ}$$



The location of point P can be expressed as **either** P(x=-3, y=-3, z=2) or P($\rho = 3\sqrt{2}$, $\phi = 225^{\circ}$, $z = 2$).

We can also perform a **coordinate transformation** on a **scalar field**. For **example**, consider the scalar field (i.e., scalar function):

$$g(\rho,\phi,z) = \rho^3 \sin\phi\, z$$

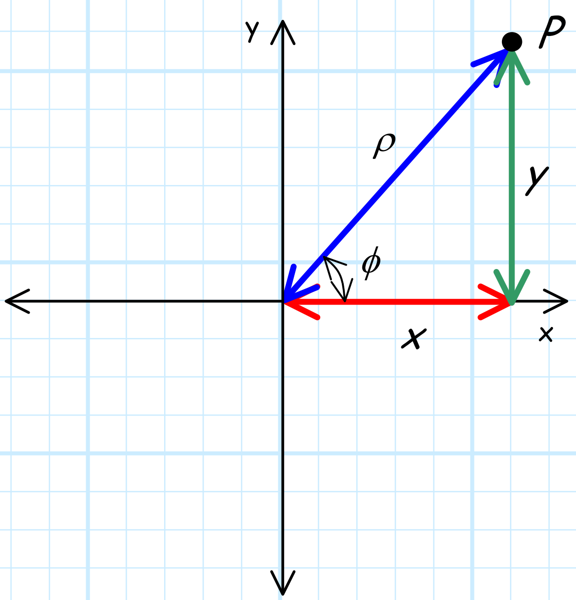
Lets try to **rewrite** this function in terms of **Cartesian** coordinates. We first note that since $\rho = \sqrt{x^2 + y^2}$,

$$\rho^3 = \left(x^2 + y^2\right)^{3/2}$$

Now, what about **sinϕ**? We know that $\phi = \tan^{-1}[y/x]$, thus we might be tempted to write:

$$\sin\phi = \sin\left[\tan^{-1}\left[\frac{y}{x}\right]\right]$$

Although **technically** correct, this is one **ugly** expression. We can instead turn to one of the **very important right triangles** that we discussed earlier:



From **this** triangle, it is apparent that:

$$\sin\phi =$$

As a result, the scalar field can be written in **Cartesian** coordinates as:

$$g(x,y,z) = \left(x^2 + y^2\right)^{3/2} \frac{y}{\sqrt{x^2 + y^2}} \, z$$
$$= \left(x^2 + y^2\right) yz$$

Remember, although the scalar fields:

$$g(x,y,z) = \left(x^2 + y^2\right) yz$$

and:

$$g(\rho,\phi,z) = \rho^3 \sin\phi \, z$$

look very different, they are in fact **exactly** the same functions—only expressed using different **coordinate variables**.

For **example**, if you **evaluate** each of the scalar fields at the **point** described earlier in the handout, you will get **exactly the same** result!

$$g(x=-3, y=-3, z=2) = -108$$

$$g(\rho = 3\sqrt{2}, \phi = 225^\circ, z = 2) = -108$$

Base Vectors

Q: *You said earlier that **vector** quantities (either discrete or field) have **both** and magnitude and **direction**. But how do we **specify** direction in 3-D space? Do we use **coordinate** values (e.g., x, y, z)??*

A: It is very important that you understand that **coordinates only** allow us to specify **position** in 3-D space. They **cannot** be used to specify **direction**!

The most convenient way for us to specify the direction of a vector quantity is by using a well-defined **orthornormal set** of vectors known as **base vectors**.

Recall that an orthonormal set of vectors, say $\hat{a}_1, \hat{a}_2, \hat{a}_3$, have the following properties:

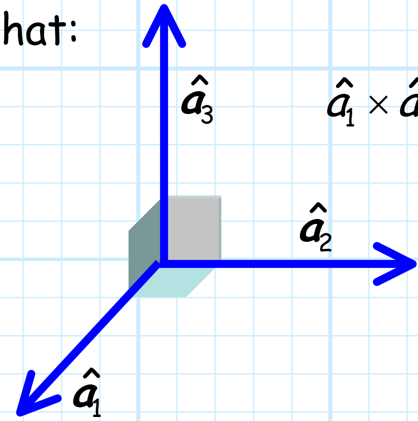
1. Each vector is a **unit** vector:

$$\hat{a}_1 \cdot \hat{a}_1 = \hat{a}_2 \cdot \hat{a}_2 = \hat{a}_3 \cdot \hat{a}_3 = 1$$

2. Each vector is mutually **orthogonal**:

$$\hat{a}_1 \cdot \hat{a}_2 = \hat{a}_2 \cdot \hat{a}_3 = \hat{a}_3 \cdot \hat{a}_1 = 0$$

Additionally, a set of base vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ must be arranged such that:

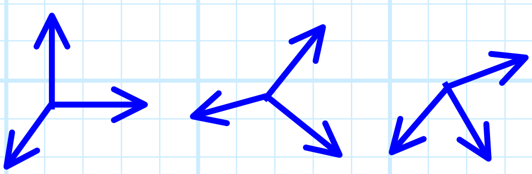


$$\hat{a}_1 \times \hat{a}_2 = \hat{a}_3, \quad \hat{a}_2 \times \hat{a}_3 = \hat{a}_1, \quad \hat{a}_3 \times \hat{a}_1 = \hat{a}_2$$

An orthonormal set with this property is known as a **right-handed** system.

> **All** base vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ must form a **right-handed, orthonormal** set.

Recall that we use **unit vectors** to define **direction**. Thus, a set of base vectors defines three distinct directions in our 3-D space!



Q: *But, **what three** directions do we use?? I remember that you said there are an **infinite** number of possible **orientations** of an orthonormal set!!*

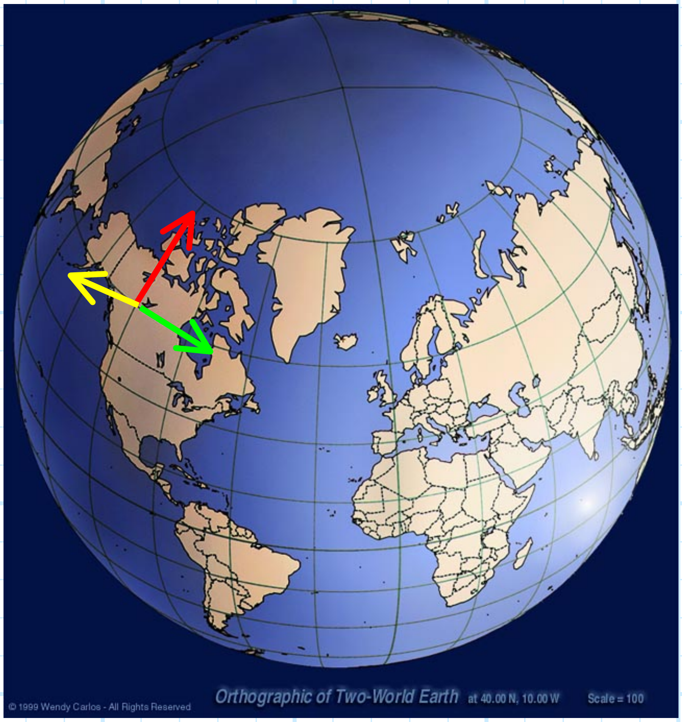
A: We will define several systematic, mathematically **precise methods** for defining the orientation of base vectors. Generally speaking, we will find that the orientation of these base vectors will **not be fixed**, but will in fact vary with **position** in space (i.e., as a function of coordinate values)!

Essentially, we will define at **each** and every point in space a **different** set of basis vectors, which can be used to uniquely define the direction of any vector quantity **at that point**!

Q: *Good golly! Defining a **different** set of base vectors for **every** point in space just seems **dad-gum confusing**. Why can't we just **fix** a set of base vectors such that their orientation is the **same** at **all** points in space?*

A: We will in fact study **one** method for defining base vectors that **does** in fact result in an othonormal set whose orientation is **fixed**—the same at **all** points in space (Cartesian base vectors).

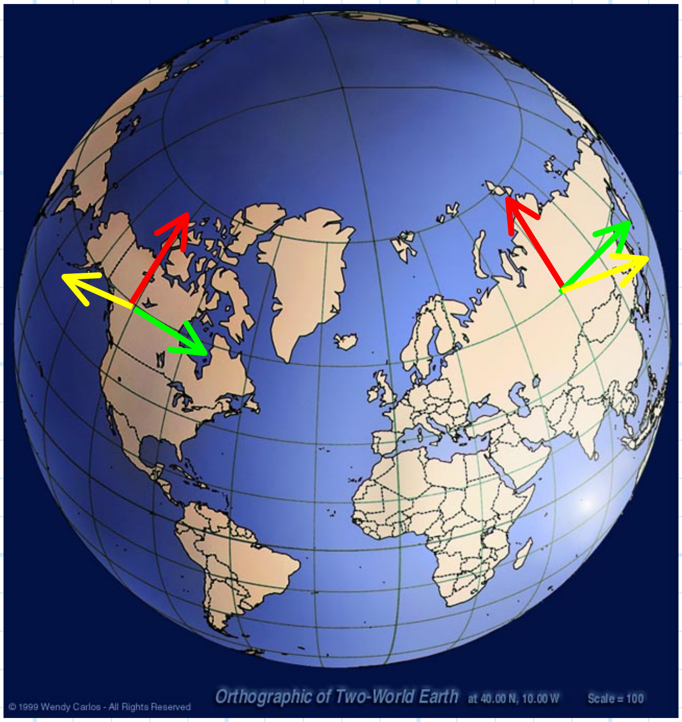
However, we will study **two other** methods where the orientation of base vectors is **different** at all points in space (spherical and cylindrical base vectors). We use these two methods to define base vectors because for **many** physical problems, it is actually **easier** and **wiser** to do so!



For example, consider how we define direction on **Earth**: North/South, East/West, Up/Down.

Each of these directions can be represented by a **unit vector**, and the three unit vectors together form a set of **base vectors**.

Think about, however, how these base vectors are oriented! Since we live on the surface of a **sphere** (i.e., the Earth), it makes sense for us to orient the base vectors with **respect to the spherical surface**.



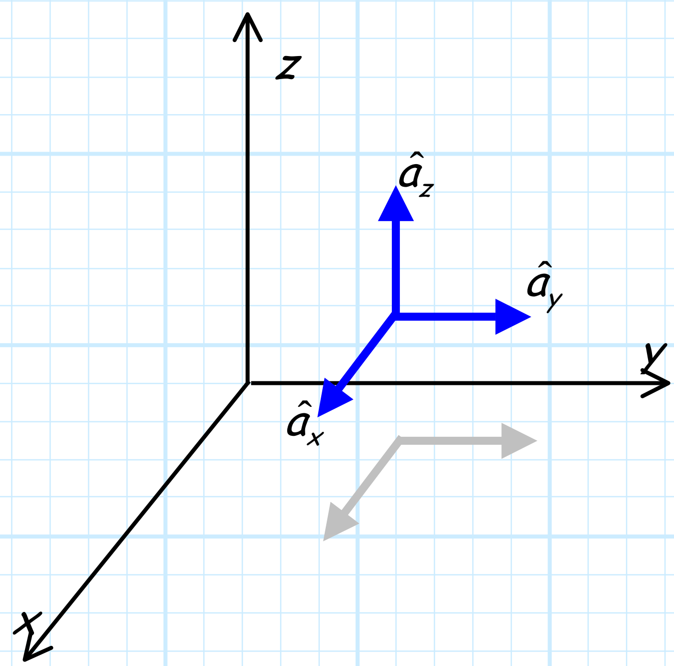
What this means, of course, is that **each location** on the Earth will orient its "base vectors" differently. This orientation is thus **different** for every point on Earth—a method that makes **perfect sense**!

Cartesian Base Vectors

As the name implies, the Cartesian base **vectors** are related to the Cartesian **coordinates**.

Specifically, the unit vector $\hat{a}_x$ points in the **direction of increasing** ***x***. In other words, it points away from the *y-z* (x=0) plane.

Similarly, $\hat{a}_y$ and $\hat{a}_z$ point in the direction of **increasing** *y* and *z*, respectfully.



We said that the directions of base vectors **generally** vary with location in space—Cartesian base vectors are the **exception**! Their directions are the same **regardless** of where you are in space.

Vector Expansion using Base Vectors

Having defined an orthonormal set of base vectors, we can express **any** vector in terms of these unit vectors:

$$\mathbf{A} = A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z$$

Note therefore that any vector can be written as a sum of three vectors!

* Each of these three vectors point in one of the **three orthogonal directions** $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.

* The **magnitude** of each of these three vectors are determined by the scalar values A_x, A_y, and A_z.

* The values A_x, A_y, and A_z are called the **scalar components** of vector **A**.

* The vectors $A_x\,\hat{a}_x$, $A_y\,\hat{a}_y$, $A_z\,\hat{a}_z$ are called the **vector components** of **A**.

Q: *What the heck are scalar the components A_x, A_y, and A_z, and how do we determine them ??*

A: Use the **dot product** to evaluate the expression above !

Begin by taking the **dot product** of the above expression with unit vector $\hat{a}_x$:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_x &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\cdot\hat{a}_x \\ &= A_x\,\hat{a}_x\cdot\hat{a}_x + A_y\,\hat{a}_y\cdot\hat{a}_x + A_z\,\hat{a}_z\cdot\hat{a}_x\end{aligned}$$

But, since the unit vectors are **orthogonal**, we know that:

$$\hat{a}_x\cdot\hat{a}_x = 1 \qquad \hat{a}_y\cdot\hat{a}_x = 0 \qquad \hat{a}_z\cdot\hat{a}_x = 0$$

Thus, the expression above becomes:

$$A_x = \mathbf{A}\cdot\hat{a}_x$$

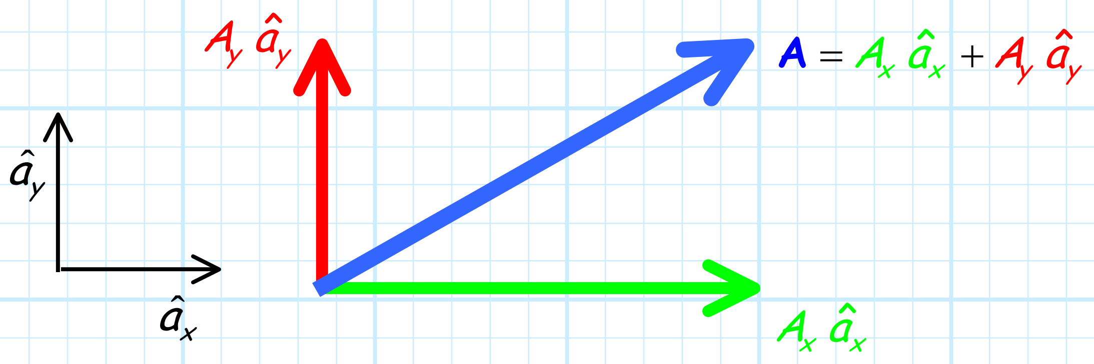
In other words, the scalar component A_x is just the value of the **dot product** of vector **A** and base vector $\hat{a}_x$. Similarly, we find that:

$$A_y = \mathbf{A}\cdot\hat{a}_y \quad \text{and} \quad A_z = \mathbf{A}\cdot\hat{a}_z$$

Thus, **any vector** can be expressed specifically as:

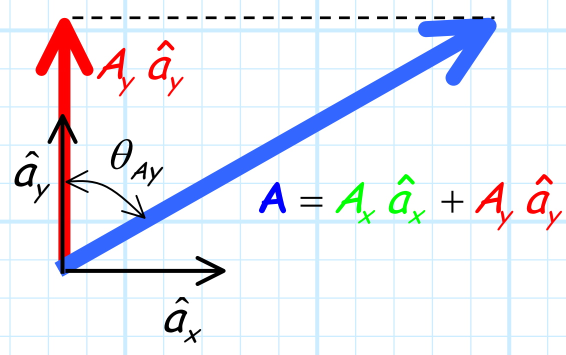
$$\begin{aligned}\mathbf{A} &= \left(\mathbf{A}\cdot\hat{a}_x\right)\hat{a}_x + \left(\mathbf{A}\cdot\hat{a}_y\right)\hat{a}_y + \left(\mathbf{A}\cdot\hat{a}_z\right)\hat{a}_z \\ &= A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\end{aligned}$$

We can demonstrate this vector expression geometrically.

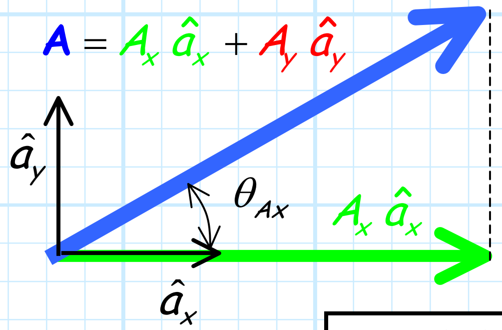


Note the length (i.e., magnitude) of vector **A** can be related to the length of vector $A_y\,\hat{a}_y$ using trigonometry:

$$A_y = |\mathbf{A}| \cos\theta_{Ay}$$



Likewise, we find that the scalar component A_x is related to $|\mathbf{A}|$ as:



$$A_x = |\mathbf{A}|\cos\theta_{Ax}$$

From this geometric interpretation, we can see why we often refer to the scalar component A_x as the **scalar projection** of vector **A onto** vector (direction) $\hat{a}_x$.

Likewise, we often refer to the vector component $A_x\,\hat{a}_x$ as the **vector projection** of vector **A onto** vector (direction) $\hat{a}_x$.

*As **you** may have already noticed, the **scalar component** A_x, which we determined geometrically, can likewise be expressed in terms of a **dot product**!*

$$\begin{aligned} A_x &= |\mathbf{A}|\cos\theta_{Ax} \\ &= |\mathbf{A}|\,|\hat{a}_x|\cos\theta_{Ax} \\ &= \mathbf{A}\cdot\hat{a}_x \end{aligned}$$

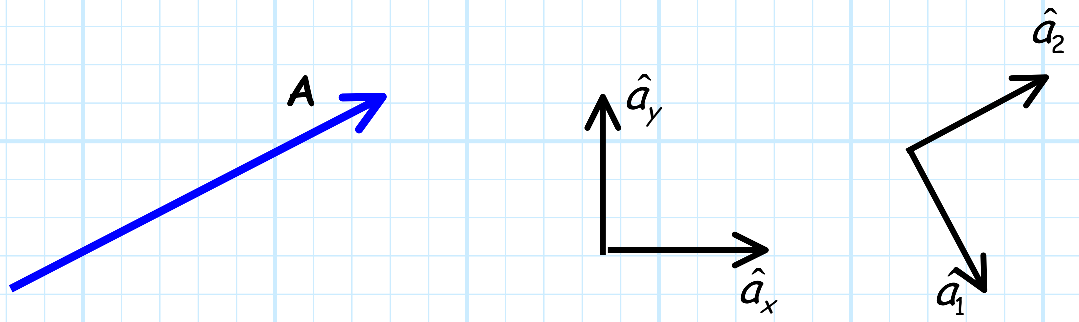
Accordingly, we find that the scalar component of vector A are determined by "doting" vector **A** with each of the three base vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$A_x = \mathbf{A} \cdot \hat{a}_x$$
$$A_y = \mathbf{A} \cdot \hat{a}_y$$
$$A_z = \mathbf{A} \cdot \hat{a}_z$$

Said another way, we **project** vector **A** onto the directions $\hat{a}_x, \hat{a}_y, \hat{a}_z$. Either way, the result is the same as determined earlier: **every** vector **A** can be expressed as a **sum** of **three** orthogonal **components**:

$$\begin{aligned}\mathbf{A} &= (\mathbf{A} \cdot \hat{a}_x)\hat{a}_x + (\mathbf{A} \cdot \hat{a}_y)\hat{a}_y + (\mathbf{A} \cdot \hat{a}_z)\hat{a}_z \\ &= A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\end{aligned}$$

For example, consider a vector **A**, along with **two different** sets of orthonormal base vectors:

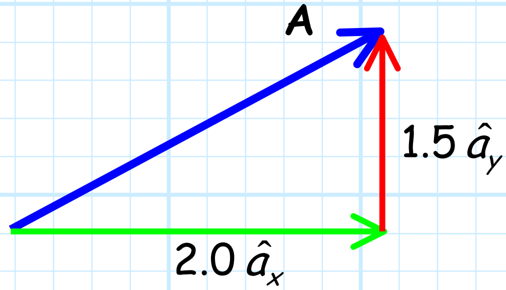


The **scalar components** of vector **A**, in the direction of each base vector are:

$$A_x = \mathbf{A} \cdot \hat{a}_x = 2.0 \qquad A_1 = \mathbf{A} \cdot \hat{a}_1 = 0.0$$
$$A_y = \mathbf{A} \cdot \hat{a}_y = 1.5 \qquad A_2 = \mathbf{A} \cdot \hat{a}_2 = 2.5$$
$$A_z = \mathbf{A} \cdot \hat{a}_z = 0.0 \qquad A_3 = \mathbf{A} \cdot \hat{a}_3 = 0.0$$

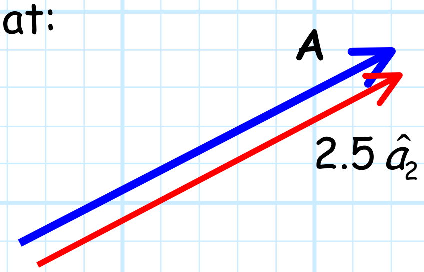
Using the **first** set of base vectors, we can write the vector **A** as:

$$\begin{aligned}\mathbf{A} &= A_x\, \hat{a}_x + A_y\, \hat{a}_y + A_z\, \hat{a}_z \\ &= 2.0\, \hat{a}_x + 1.5\, \hat{a}_y\end{aligned}$$



Or, using the **second** set, we find that:

$$\begin{aligned}\mathbf{A} &= A_1\, \hat{a}_1 + A_2\, \hat{a}_2 + A_3\, \hat{a}_3 \\ &= 2.5\, \hat{a}_2\end{aligned}$$



It is **very** important to realize that:

$$\mathbf{A} = 2.0\, \hat{a}_x + 1.5\, \hat{a}_y = 2.5\, \hat{a}_2$$

In other words, both expressions represent **exactly** the same vector! The difference in the representations is a result of using **different base vectors**, not because vector **A** is somehow "different" for each representation.

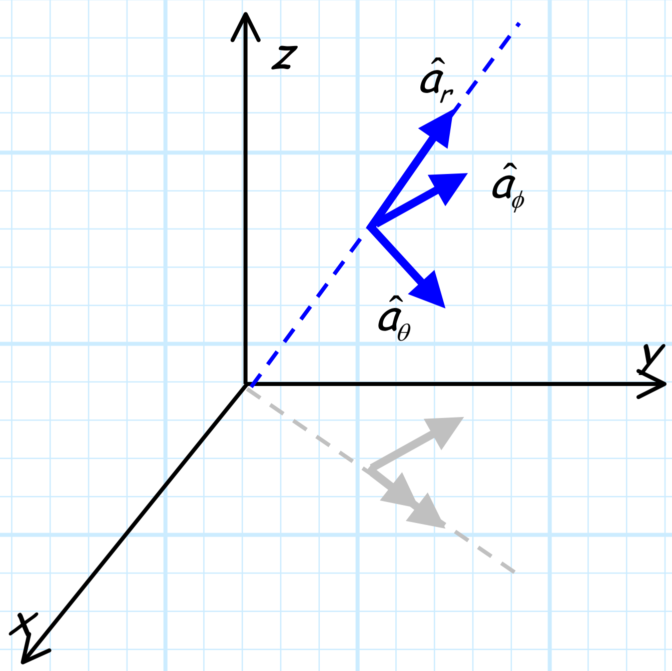
Spherical Base Vectors

Spherical base vectors are the "natural" base vectors of a **sphere**.

$\hat{a}_r$ points in the direction of **increasing** r. In other words $\hat{a}_r$ points **away from the origin**. This is analogous to the direction we call **up**.

$\hat{a}_\theta$ points in the direction of **increasing** θ. This is analogous to the direction we call **south.**

$\hat{a}_\phi$ points in the direction of increasing ϕ. This is analogous to the direction we call **east.**



IMPORTANT NOTE: The directions of spherical base vectors are **dependent on position**. First you must determine **where** you are in space (using coordinate values), **then** you can define the directions of $\hat{a}_r$, $\hat{a}_\theta$, $\hat{a}_\phi$.

Note **Cartesian** base vectors are **special**, in that their directions are **independent** of location—they have the same directions throughout all space.

Thus, it is helpful to define spherical base vectors **in terms of** Cartesian base vectors. It can be shown that:

$$\begin{aligned} \hat{a}_r \cdot \hat{a}_x &= \sin\theta\cos\phi & \hat{a}_\theta \cdot \hat{a}_x &= \cos\theta\cos\phi & \hat{a}_\phi \cdot \hat{a}_x &= -\sin\phi \\ \hat{a}_r \cdot \hat{a}_y &= \sin\theta\sin\phi & \hat{a}_\theta \cdot \hat{a}_y &= \cos\theta\sin\phi & \hat{a}_\phi \cdot \hat{a}_y &= \cos\phi \\ \hat{a}_r \cdot \hat{a}_z &= \cos\theta & \hat{a}_\theta \cdot \hat{a}_z &= -\sin\theta & \hat{a}_\phi \cdot \hat{a}_z &= 0 \end{aligned}$$

Recall that **any** vector **A** can be written as:

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{a}_x\right)\hat{a}_x + \left(\mathbf{A}\cdot\hat{a}_y\right)\hat{a}_y + \left(\mathbf{A}\cdot\hat{a}_z\right)\hat{a}_z .$$

Therefore, we can write unit vector $\hat{a}_r$ as, for example:

$$\begin{aligned} \hat{a}_r &= \left(\hat{a}_r\cdot\hat{a}_x\right)\hat{a}_x + \left(\hat{a}_r\cdot\hat{a}_y\right)\hat{a}_y + \left(\hat{a}_r\cdot\hat{a}_z\right)\hat{a}_z \\ &= \sin\theta\cos\phi\ \hat{a}_x + \sin\theta\sin\phi\ \hat{a}_y + \cos\theta\ \hat{a}_z \end{aligned}$$

This result explicitly shows that $\hat{a}_r$ is a function of θ and ϕ.

For **example**, at the point in space $r = 7.239$, $\theta = 90^\circ$ and $\phi = 0^\circ$, we find that $\hat{a}_r = \hat{a}_x$. In other words, at this point in space, the direction $\hat{a}_r$ points in the ***x***-direction.

Or, at the point in space $r = 2.735$, $\theta = 90^\circ$ and $\phi = 90^\circ$, we find that $\hat{a}_r = \hat{a}_y$. In other words, at this point in space, $\hat{a}_r$ points in the ***y***-direction.

Additionally, we can write $\hat{a}_\theta$ and $\hat{a}_\phi$ as:

$$\hat{a}_\theta = \left(\hat{a}_\theta \cdot \hat{a}_x\right)\hat{a}_x + \left(\hat{a}_\theta \cdot \hat{a}_y\right)\hat{a}_y + \left(\hat{a}_\theta \cdot \hat{a}_z\right)\hat{a}_z$$

$$\hat{a}_\phi = \left(\hat{a}_\phi \cdot \hat{a}_x\right)\hat{a}_x + \left(\hat{a}_\phi \cdot \hat{a}_y\right)\hat{a}_y + \left(\hat{a}_\phi \cdot \hat{a}_z\right)\hat{a}_z$$

Alternatively, we can write **Cartesian** base vectors in terms of spherical base vectors, i.e.,

$$\hat{a}_x = \left(\hat{a}_x \cdot \hat{a}_r\right)\hat{a}_r + \left(\hat{a}_x \cdot \hat{a}_\theta\right)\hat{a}_\theta + \left(\hat{a}_x \cdot \hat{a}_\phi\right)\hat{a}_\phi$$

$$\hat{a}_y = \left(\hat{a}_y \cdot \hat{a}_r\right)\hat{a}_r + \left(\hat{a}_y \cdot \hat{a}_\theta\right)\hat{a}_\theta + \left(\hat{a}_y \cdot \hat{a}_\phi\right)\hat{a}_\phi$$

$$\hat{a}_z = \left(\hat{a}_z \cdot \hat{a}_r\right)\hat{a}_r + \left(\hat{a}_z \cdot \hat{a}_\theta\right)\hat{a}_\theta + \left(\hat{a}_z \cdot \hat{a}_\phi\right)\hat{a}_\phi$$

Using the **table** on the previous page, we can insert the result of each dot product to express each base vector in terms of **spherical coordinates**!

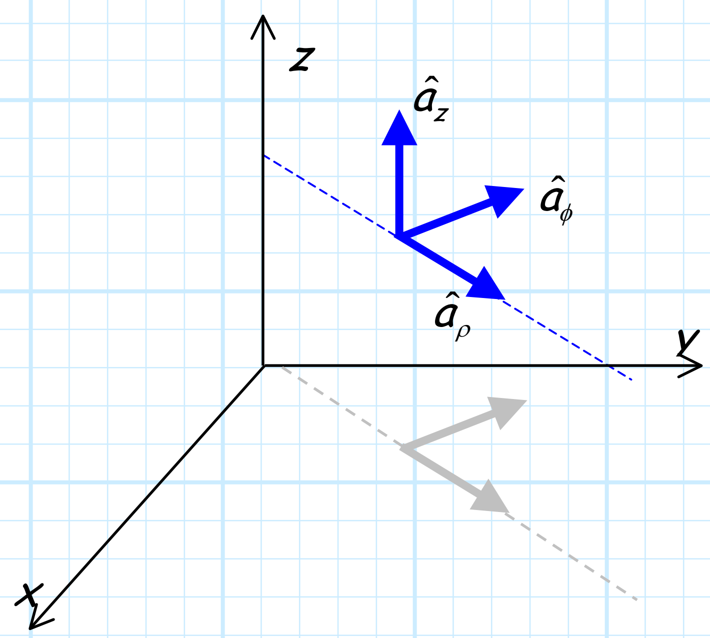
Cylindrical Base Vectors

Cylindrical base vectors are the **natural** base vectors of a **cylinder**.

$\hat{a}_\rho$ points in the direction of **increasing** ρ. In other words, $\hat{a}_\rho$ points **away from the *z*-axis**.

$\hat{a}_\phi$ points in the direction of **increasing** ϕ. This is precisely the **same** base vector we described for **spherical** base vectors.

$\hat{a}_z$ points in the direction of **increasing *z***. This is precisely the **same** base vector we described for **Cartesian** base vectors.



It is evident, that like spherical base vectors, the cylindrical base vectors are **dependent on position**. A vector that points **away** from the z-axis (e.g., $\hat{a}_\rho$), will point in a direction that is **dependent** on where we are in space!

We can express cylindrical base vectors in terms of **Cartesian** base vectors. First, we find that:

$$\begin{array}{lll} \hat{a}_\rho \cdot \hat{a}_x = \cos\phi & \hat{a}_\phi \cdot \hat{a}_x = -\sin\phi & \hat{a}_z \cdot \hat{a}_x = 0 \\ \hat{a}_\rho \cdot \hat{a}_y = \sin\phi & \hat{a}_\phi \cdot \hat{a}_y = \cos\phi & \hat{a}_z \cdot \hat{a}_y = 0 \\ \hat{a}_\rho \cdot \hat{a}_z = 0 & \hat{a}_\phi \cdot \hat{a}_z = 0 & \hat{a}_z \cdot \hat{a}_z = 1 \end{array}$$

We can use these results to write **cylindrical** base vectors in terms of **Cartesian** base vectors, or vice versa!

For **example**,

$$\begin{aligned} \hat{a}_p &= \left(\hat{a}_p \cdot \hat{a}_x \right) \hat{a}_x + \left(\hat{a}_p \cdot \hat{a}_y \right) \hat{a}_y + \left(\hat{a}_p \cdot \hat{a}_z \right) \hat{a}_z \\ &= \cos\phi \, \hat{a}_x + \sin\phi \, \hat{a}_y \end{aligned}$$

or,

$$\begin{aligned} \hat{a}_x &= \left(\hat{a}_x \cdot \hat{a}_\rho \right) \hat{a}_\rho + \left(\hat{a}_x \cdot \hat{a}_\phi \right) \hat{a}_\phi + \left(\hat{a}_x \cdot \hat{a}_z \right) \hat{a}_z \\ &= \cos\phi \, \hat{a}_\rho - \sin\phi \, \hat{a}_\phi \end{aligned}$$

Finally, we can write **cylindrical** base vectors in terms of **spherical** base vectors, or vice versa, using the following relationships:

$$
\begin{array}{lll}
\hat{a}_\rho \cdot \hat{a}_r = \sin\theta & \hat{a}_\phi \cdot \hat{a}_r = 0 & \hat{a}_z \cdot \hat{a}_r = \cos\theta \\
\hat{a}_\rho \cdot \hat{a}_\theta = \cos\theta & \hat{a}_\phi \cdot \hat{a}_\theta = 0 & \hat{a}_z \cdot \hat{a}_\theta = -\sin\theta \\
\hat{a}_\rho \cdot \hat{a}_\phi = 0 & \hat{a}_\phi \cdot \hat{a}_\phi = 1 & \hat{a}_z \cdot \hat{a}_\phi = 0
\end{array}
$$

e.g.,

$$
\begin{aligned}
\hat{a}_p &= \left(\hat{a}_p \cdot \hat{a}_r \right)\hat{a}_r + \left(\hat{a}_p \cdot \hat{a}_\theta \right)\hat{a}_\theta + \left(\hat{a}_p \cdot \hat{a}_\phi \right)\hat{a}_\phi \\
&= \sin\theta\, \hat{a}_r + \cos\theta\, \hat{a}_\theta
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_\theta &= \left(\hat{a}_\theta \cdot \hat{a}_\rho \right)\hat{a}_\rho + \left(\hat{a}_\theta \cdot \hat{a}_\phi \right)\hat{a}_\phi + \left(\hat{a}_\theta \cdot \hat{a}_z \right)\hat{a}_z \\
&= \cos\theta\, \hat{a}_\rho - \sin\theta\, \hat{a}_z
\end{aligned}
$$

Vector Algebra using Orthonormal Base Vectors

Q: *Just why do we express a vector in terms of 3 orthonormal base vectors? Doesn't this just make things even more complicated ??*

A: Actually, it makes things **much** simpler. The **evaluation** of vector operations such as addition, subtraction, multiplication, dot product, and cross product all become straightforward **if** all vectors are expressed using the **same** set of base vectors.

Consider two vectors **A** and **B**, each expressed using the same set of base vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$\mathbf{A} = A_x\, \hat{a}_x + A_y\, \hat{a}_y + A_z\, \hat{a}_z$$

$$\mathbf{B} = B_x\, \hat{a}_x + B_y\, \hat{a}_y + B_z\, \hat{a}_z$$

1. *Addition and Subtraction*

If we **add** these two vectors together, we find:

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)+\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x + B_x\,\hat{a}_x + A_y\,\hat{a}_y + B_y\,\hat{a}_y + A_z\,\hat{a}_z + B_z\,\hat{a}_z\\
&= \left(A_x + B_x\right)\hat{a}_x + \left(A_y + B_y\right)\hat{a}_y + \left(A_z + B_z\right)\hat{a}_z
\end{aligned}
$$

In other words, each component of the **sum** of two vectors is equal to the sum of each **component**.

Similarly, we find for **subtraction**:

$$
\begin{aligned}
\mathbf{A}-\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)-\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x - B_x\,\hat{a}_x + A_y\,\hat{a}_y - B_y\,\hat{a}_y + A_z\,\hat{a}_z - B_z\,\hat{a}_z\\
&= \left(A_x - B_x\right)\hat{a}_x + \left(A_y - B_y\right)\hat{a}_y + \left(A_z - B_z\right)\hat{a}_z
\end{aligned}
$$

2. *Vector/Scalar Multiplication*

Say we multiply a scalar a and a vector **B**, i.e., $a\,\mathbf{B}$:

$$
\begin{aligned}
a\mathbf{B} &= a\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= aB_x\,\hat{a}_x + aB_y\,\hat{a}_y + aB_z\,\hat{a}_z\\
&= \left(aB_x\right)\hat{a}_x + \left(aB_y\right)\hat{a}_y + \left(aB_z\right)\hat{a}_z
\end{aligned}
$$

In other words, each component of the product of a scalar and a vector are equal to the product of the scalar and each component.

3. Dot Product

Say we take the **dot product** of **A** and **B**:

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_y\,\hat{a}_y\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_z\,\hat{a}_z\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,B_x\left(\hat{a}_x\cdot\hat{a}_x\right) + A_x\,B_y\left(\hat{a}_x\cdot\hat{a}_y\right) + A_x\,B_z\left(\hat{a}_x\cdot\hat{a}_z\right)\\
&\quad + A_y\,B_x\left(\hat{a}_y\cdot\hat{a}_x\right) + A_y\,B_y\left(\hat{a}_y\cdot\hat{a}_y\right) + A_y\,B_y\left(\hat{a}_y\cdot\hat{a}_z\right)\\
&\quad + A_z\,B_x\left(\hat{a}_z\cdot\hat{a}_x\right) + A_z\,B_y\left(\hat{a}_z\cdot\hat{a}_y\right) + A_z\,B_z\left(\hat{a}_z\cdot\hat{a}_z\right)
\end{aligned}
$$

A: Be patient! Recall that these are **orthonormal** base vectors, therefore:

$$
\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1 \quad \text{and} \quad \hat{a}_x\cdot\hat{a}_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0
$$

As a result, our **dot product** expression reduces to this simple expression:

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

We can apply this to the expression for determining the **magnitude** of a vector:

$$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2$$

Therefore:

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

For example, consider a previous handout, where we expressed a vector using two different sets of basis vectors:

$$\mathbf{A} = 2.0\hat{a}_x + 1.5\hat{a}_y$$

or,

$$\mathbf{A} = 2.5\hat{b}_y$$

Therefore, the magnitude of **A** is determined to be:

$$|\mathbf{A}| = \sqrt{1.5^2 + 2.0^2} = \sqrt{6.25} = 2.5$$

or,

$$|\mathbf{A}| = \sqrt{2.5^2} = \sqrt{6.25} = 2.5$$

Q: *Hey! We get the **same** answer from both expressions; is this a **coincidence** ?*

A: No! Remember, both expressions represent the **same** vector, only using different sets of base vectors. The magnitude of vector **A** is 2.5, **regardless** of how we choose to express **A**.

4. *Cross Product*

Now lets take the cross product **A**x**B**:

$$\begin{aligned}
\mathbf{A}\mathsf{x}\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_y\,\hat{a}_y \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_z\,\hat{a}_z \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_xB_x\left(\hat{a}_x\times\hat{a}_x\right) + A_xB_y\left(\hat{a}_x\times\hat{a}_y\right) + A_xB_z\left(\hat{a}_x\times\hat{a}_z\right)\\
&\quad + A_yB_x\left(\hat{a}_y\times\hat{a}_x\right) + A_yB_y\left(\hat{a}_y\times\hat{a}_y\right) + A_yB_z\left(\hat{a}_y\times\hat{a}_z\right)\\
&\quad + A_zB_x\left(\hat{a}_z\times\hat{a}_x\right) + A_zB_y\left(\hat{a}_z\times\hat{a}_y\right) + A_zB_z\left(\hat{a}_z\times\hat{a}_z\right)
\end{aligned}$$

Remember, we know that:

$$\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

also, since base vectors form a **right-handed** system:

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z \qquad \hat{a}_y \times \hat{a}_z = \hat{a}_x \qquad \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

Remember also that **A**x**B** = -(**B**x**A**), therefore:

$$\hat{a}_y \times \hat{a}_x = -\hat{a}_z \qquad \hat{a}_z \times \hat{a}_y = -\hat{a}_x \qquad \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

Combining all the equations above, we get:

$$\mathbf{A}\times\mathbf{B} = (A_y B_z - A_z B_y)\,\hat{a}_x + (A_z B_x - A_x B_z)\,\hat{a}_y + (A_x B_y - A_y B_x)\,\hat{a}_z$$

5. *Triple Product*

Combining the results of the dot product and the cross product, we find that the **triple product** can be expressed as:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \left(A_x B_y C_z + A_y B_z C_x + A_z B_x C_y\right) - \left(A_x B_z C_y + A_y B_x C_z + A_z B_y C_x\right)$$

IMPORTANT NOTES:

*In **addition** to all that we have discussed here, it is **critical** that you understand the following points about vector algebra using orthonormal base vectors!*

* The results provided in this handout were given for **Cartesian** base vectors ($\hat{a}_x, \hat{a}_y, \hat{a}_z$). However, they are equally valid for **any** right-handed set of base vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ (e.g., $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ or $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$).

* These results are **algorithms** for evaluating various vector algebraic operations. They are **not** definitions of the operations. The **definitions** of these operations were covered in **Section 2-3**.

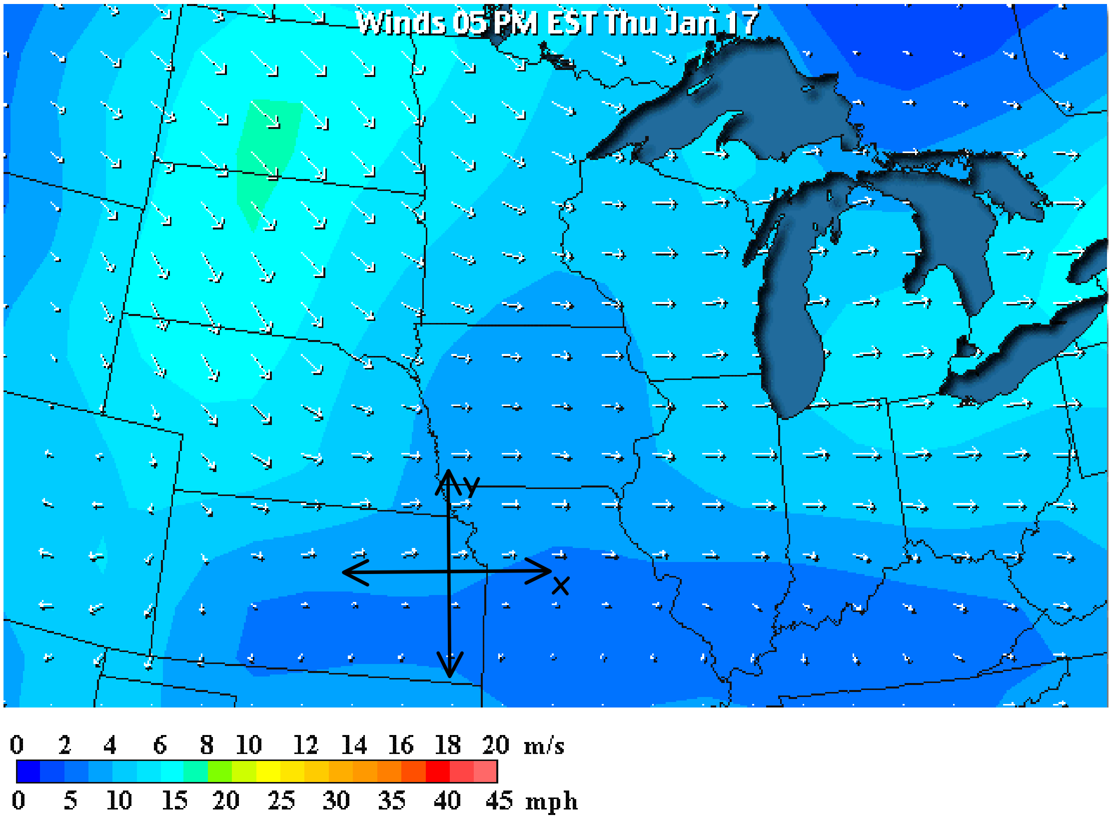
* The scalar components A_x, A_y, and A_z represent **either** discrete scalar (e.g., $A_x = 4.2$) **or** scalar field quantities (e.g., $A_\theta = r^2 \sin\theta \cos\phi$.

Vector Fields

Base vectors give us a convenient way to express **vector fields**!

You will recall that a **vector field** is a vector quantity that is a **function** of other scalar values. In this class, we will study vector fields that are a function of **position** (e.g., $\mathbf{A}(x,y,z)$).

We earlier considered an **example** of a vector field of this type: the wind **velocity** $\mathbf{v}(x,y)$ across the upper Midwest.



When we express a vector field using orthonormal **base vectors**, the **scalar component** of each direction is a **scalar field**—a scalar function of position!

In other words, a **vector field** can have the form:

$$\mathbf{A}(x,y,z) = A_x(x,y,z)\,\hat{a}_x + A_y(x,y,z)\,\hat{a}_y + A_z(x,y,z)\,\hat{a}_z$$

We therefore can express a **vector** field $\mathbf{A}(x,y,z)$ in terms of **3 scalar** fields: $A_x(x,y,z)$, $A_y(x,y,z)$, and $A_z(x,y,z)$, which express each of the 3 scalar **components** as a **function** of position (x,y,z).

For example, we might encounter this **vector field**:

$$\mathbf{A}(x,y,z) = (x^2 + y^2)\,\hat{a}_x + \frac{xz}{y}\,\hat{a}_y + (3-y)\,\hat{a}_z$$

In this case it is evident that:

$$A_x(x,y,z) = (x^2 + y^2)$$

$$A_y(x,y,z) = \frac{xz}{y}$$

$$A_z(x,y,z) = (3-y)$$

The vector algebraic rules that we discussed in previous handouts are just as **valid** for vector **fields** and scalar **field** components as they are for **discrete** vectors and **discrete** scalar components.

For example, consider these two **vector fields**, expressed in terms of orthonormal **base vectors** $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$\mathbf{A}(x,y,z) = y^2\,\hat{a}_x + (x-z)\,\hat{a}_y + \frac{y}{z}\,\hat{a}_z$$

$$\mathbf{B}(x,y,z) = (x+2)\,\hat{a}_x + z\,\hat{a}_y + xyz\,\hat{a}_z$$

The **dot product** of these two vector fields is a **scalar field**:

$$\begin{aligned}\mathbf{A}(x,y,z)\cdot\mathbf{B}(x,y,z) &= A_x\,B_x + A_y\,B_y + A_z\,B_z \\ &= y^2(x+2) + (xz - z^2) + xy^2\end{aligned}$$

Likewise, the **sum** of these two vector fields is a **vector field**:

$$\begin{aligned}\mathbf{A}(x,y,z)+\mathbf{B}(x,y,z) &= (A_x + B_x)\,\hat{a}_x + (A_y + B_y)\,\hat{a}_y + (A_z + B_z)\,\hat{a}_z \\ &= (y^2 + x + 2)\,\hat{a}_x + x\,\hat{a}_y + \frac{y(xz^2+1)}{z}\,\hat{a}_z\end{aligned}$$

Note the example vector fields we have shown here are a function of **spatial** coordinates **only**. In other words, the vector field is **constant** with respect to **time**—the discrete vector quantity at any and every point in space **never changes** its magnitude or direction.

However, we find that many (if not most) vector fields found in nature **do** change with respect to both spatial position **and** time.

Thus, we often discover that vector fields must be written as variables of three spatial coordinates, as well as a **time** variable t!

For example:

$$\mathbf{A}(x,y,z,t) = (x^2 + y^2)t\ \hat{a}_x + \frac{xz}{y}t^2\ \hat{a}_y + (3 - y + 4t)\ \hat{a}_z$$

* A vector field that **changes** with respect to time is known as a **dynamic** vector field.

* A vector field that is **constant** with respect to time is known as a **static** vector field.

Example: Expressing Vector Fields with Coordinate Systems

Consider the vector **field**:

$$\mathbf{A} = xz\,\hat{a}_x + \left(x^2 + y^2\right)\hat{a}_y + \left(\frac{x}{z}\right)\hat{a}_z$$

Let's try to accomplish **three** things:

1. Express **A** using **spherical** coordinates and **Cartesian** base vectors.

2. Express **A** using **Cartesian** coordinates and **spherical** base vectors.

3. Express **A** using **cylindrical** coordinates and **cylindrical** base vectors.

1. The vector field is **already** expressed with Cartesian **base vectors**, therefore we only need to change the Cartesian **coordinates** in each **scalar component** into spherical coordinates.

The **scalar component** of **A** in the *x*-direction is:

$$\begin{aligned} A_x &= xz \\ &= (r\sin\theta\cos\phi)(r\cos\theta) \\ &= r^2\sin\theta\cos\theta\cos\phi \end{aligned}$$

The scalar component of **A** in the *y*-direction is:

$$\begin{aligned} A_y &= x^2 + y^2 \\ &= (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 \\ &= r^2\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) \\ &= r^2\sin^2\theta \end{aligned}$$

The scalar component of **A** in the *z*-direction is:

$$\begin{aligned} A_z &= \frac{x}{z} \\ &= \frac{r\sin\theta\cos\phi}{r\cos\theta} \\ &= \tan\theta\cos\phi \end{aligned}$$

Therefore, the vector field can be expressed using *spherical* ***coordinates*** as:

$$\mathbf{A} = r^2\sin\theta\cos\theta\cos\phi\,\hat{a}_x + r^2\sin^2\theta\,\hat{a}_y + \tan\theta\cos\phi\,\hat{a}_z$$

2. Now, let's express **A** using **spherical base vectors**. We **cannot** simply change the **coordinates** of each component. Rather, we must determine **new scalar components**, since we are using a **new** set of **base vectors**. We begin by stating:

$$\mathbf{A} = \left(\mathbf{A} \cdot \hat{a}_r\right) \hat{a}_r + \left(\mathbf{A} \cdot \hat{a}_\theta\right) \hat{a}_\theta + \left(\mathbf{A} \cdot \hat{a}_\phi\right) \hat{a}_\phi$$

The scalar component A_r is therefore:

$$\begin{aligned}
\mathbf{A} \cdot \hat{a}_r &= xz\, \hat{a}_x \cdot \hat{a}_r + \left(x^2 + y^2\right) \hat{a}_y \cdot \hat{a}_r + \left(\frac{x}{z}\right) \hat{a}_z \cdot \hat{a}_r \\
&= xz\left(\sin\theta\cos\phi\right) + \left(x^2 + y^2\right)\left(\sin\theta\sin\phi\right) + \left(\frac{x}{z}\right)\left(\cos\theta\right) \\
&= xz \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{x}{\sqrt{x^2 + y^2}} \\
&\quad + \left(x^2 + y^2\right) \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{y}{\sqrt{x^2 + y^2}} \\
&\quad + \left(\frac{x}{z}\right) \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
&= \frac{x^2 z}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\left(x^2 + y^2\right)}{\sqrt{x^2 + y^2 + z^2}} + \frac{x}{\sqrt{x^2 + y^2 + z^2}} \\
&= \frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

Likewise, the scalar component A_θ is:

$$\begin{aligned}
\mathbf{A}\cdot\hat{a}_\theta &= xz\ \hat{a}_x\cdot\hat{a}_\theta + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\theta + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\theta \\
&= xz\left(\cos\theta\cos\phi\right) + \left(x^2+y^2\right)\left(\cos\theta\sin\phi\right) - \left(\frac{x}{z}\right)\left(\sin\theta\right) \\
&= xz\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}} \\
&\quad +\left(x^2+y^2\right)\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}} \\
&\quad -\left(\frac{x}{z}\right)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{x^2z^3}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} + \frac{yz^2\left(x^2+y^2\right)}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \\
&\quad - \frac{x\left(x^2+y^2\right)}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \\
&= \frac{x^2z^3 + x^2yz^2 + y^3z - x^3 - xy^2}{z\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}
\end{aligned}$$

And finally, the scalar component A_ϕ is:

$$\begin{aligned}
\mathbf{A}\cdot\hat{a}_\phi &= xz\ \hat{a}_x\cdot\hat{a}_\phi + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\phi + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\phi \\
&= xz\left(-\sin\phi\right) + \left(x^2+y^2\right)\left(\cos\phi\right) + \left(\frac{x}{z}\right)0 \\
&= xz\frac{-y}{\sqrt{x^2+y^2}} + \left(x^2+y^2\right)\frac{x}{\sqrt{x^2+y^2}} \\
&= \frac{-xyz + x^3 + xy^2}{\sqrt{x^2+y^2}}
\end{aligned}$$

Whew! We're finished! The vector **A** is expressed using Cartesian coordinates and **spherical** base vectors as:

$$\mathbf{A} = \left(\frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{a}_r$$

$$+ \left(\frac{x^2 z^3 + x^2 y z^2 + y^3 z - x^3 - x y^2}{z\sqrt{x^2 + y^2 + z^2}\,\sqrt{x^2 + y^2}} \right) \hat{a}_\theta$$

$$+ \left(\frac{-xyz + x^3 + x y^2}{\sqrt{x^2 + y^2}} \right) \hat{a}_\phi$$

3. Now, let's write **A** in terms of cylindrical coordinates **and** cylindrical base vectors (i.e., in terms of the cylindrical coordinate **system**).

$$\mathbf{A} = \left(\mathbf{A} \cdot \hat{a}_\rho\right) \hat{a}_\rho + \left(\mathbf{A} \cdot \hat{a}_\phi\right) \hat{a}_\phi + \left(\mathbf{A} \cdot \hat{a}_z\right) \hat{a}_z$$

First, A_ρ is:

$$\begin{aligned}
\mathbf{A} \cdot \hat{a}_\rho &= xz\, \hat{a}_x \cdot \hat{a}_\rho + \left(x^2 + y^2\right) \hat{a}_y \cdot \hat{a}_\rho + \left(\frac{x}{z}\right) \hat{a}_z \cdot \hat{a}_\rho \\
&= xz\left(\cos\phi\right) + \left(x^2 + y^2\right)\left(\sin\phi\right) + \left(\frac{x}{z}\right)(0) \\
&= \rho\cos\phi\, z\left(\cos\phi\right) + \rho^2\left(\sin\phi\right) \\
&= \rho\cos^2\phi\, z + \rho^2 \sin\phi
\end{aligned}$$

And A_ϕ is:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_\phi &= xz\,\hat{a}_x\cdot\hat{a}_\phi + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_\phi + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_\phi \\ &= xz\left(-\sin\phi\right) + \left(x^2+y^2\right)\left(\cos\phi\right) + \left(\frac{x}{z}\right)(0) \\ &= -\rho\cos\phi\, z\left(\sin\phi\right) + \rho^2\left(\cos\phi\right) \\ &= \rho\cos\phi\left(\rho - z\sin\phi\right)\end{aligned}$$

And finally, A_z is:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_z &= xz\,\hat{a}_x\cdot\hat{a}_z + \left(x^2+y^2\right)\hat{a}_y\cdot\hat{a}_z + \left(\frac{x}{z}\right)\hat{a}_z\cdot\hat{a}_z \\ &= xz\left(0\right) + \left(x^2+y^2\right)\left(0\right) + \left(\frac{x}{z}\right)(1) \\ &= \left(\frac{x}{z}\right) \\ &= \frac{\rho\cos\phi}{z}\end{aligned}$$

We can therefore express the vector field **A** using **both** cylindrical coordinates and cylindrical base vectors:

$$\mathbf{A} = \left(\rho\cos^2\phi\, z + \rho^2\sin\phi\right)\hat{a}_\rho + \rho\cos\phi\left(\rho - z\sin\phi\right)\hat{a}_\phi + \left(\frac{\rho\cos\phi}{z}\right)\hat{a}_z$$

Thus, we have determined **three** possible ways (and there are many other ways!) to express the vector field **A**:

1.

$$\mathbf{A} = r^2 \sin\theta \, \cos\theta \, \cos\phi \, \hat{a}_x + r^2 \sin^2\theta \, \hat{a}_y + \tan\theta \cos\phi \, \hat{a}_z$$

2.

$$\begin{aligned}\mathbf{A} = &\left(\frac{x^2 z + x^2 y + y^3 + x}{\sqrt{x^2+y^2+z^2}}\right)\hat{a}_r \\ &+\left(\frac{x^2 z^3 + x^2 y z^2 + y^3 z - x^3 - x y^2}{z\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}}\right)\hat{a}_\theta \\ &+\left(\frac{-xyz + x^3 + xy^2}{\sqrt{x^2+y^2}}\right)\hat{a}_\phi\end{aligned}$$

3.

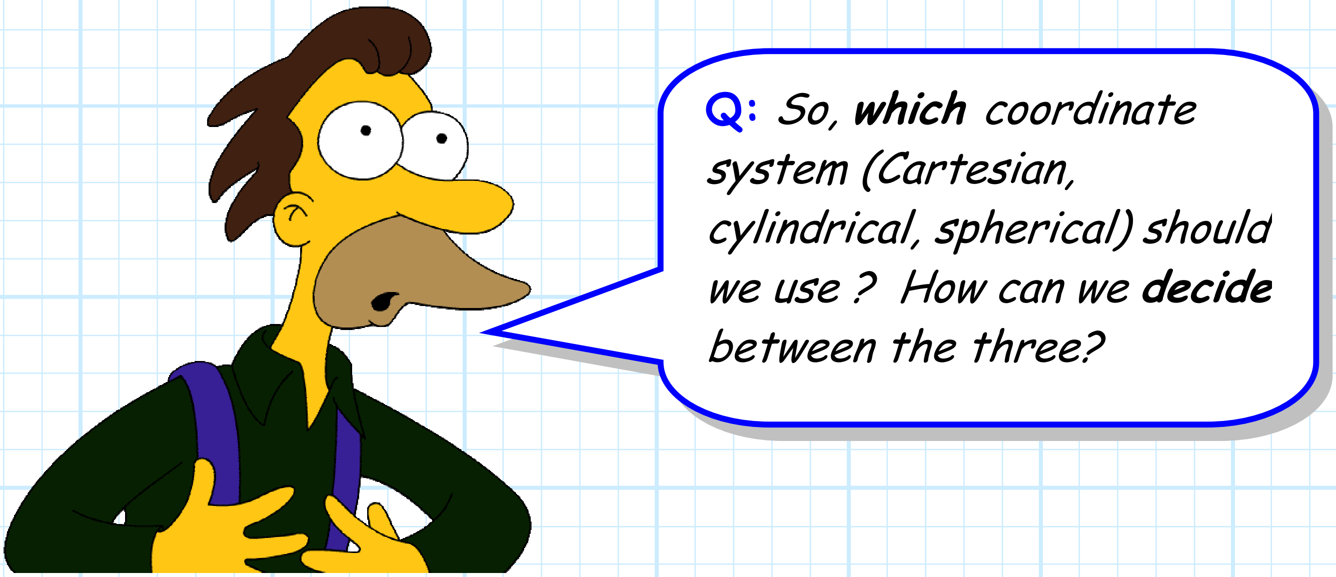
$$\mathbf{A} = \left(\rho\cos^2\phi \, z + \rho^2 \sin\phi\right)\hat{a}_\rho + \rho\cos\phi\left(\rho - z\sin\phi\right)\hat{a}_\phi + \left(\frac{\rho\cos\phi}{z}\right)\hat{a}_z$$

Please note:

* The three expressions for vector field **A** provided in this handout each **look** very different. However, they are just three different methods for describing the **same** vector field. **Any** one of the three is correct, and will result in the **same result** for any physical problem.

* We can express a vector field using **any** set of coordinate variables **and** any set of base vectors.

* Generally speaking, however, we use one coordinate **system** to describe a vector field. For example, we use **both** spherical coordinates and spherical base vectors.

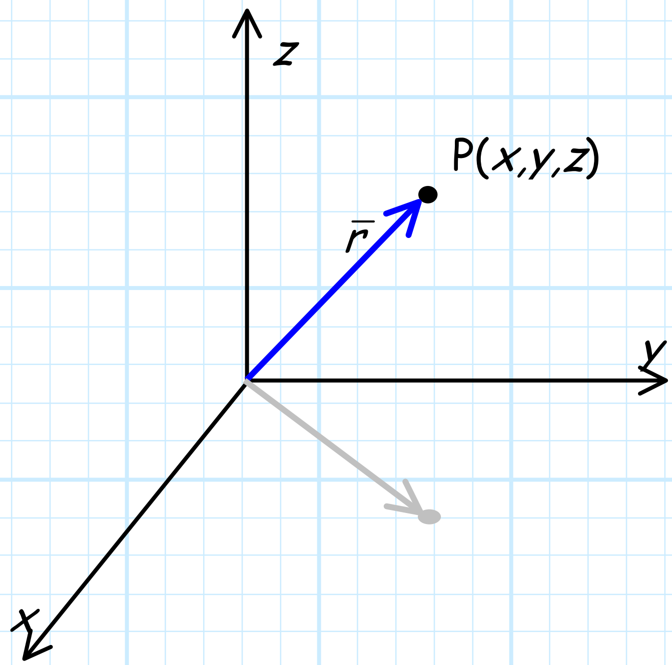


A: Ideally, we select that system that most **simplifies** the mathematics. This depends on the **physical problem** we are solving.

For example, if we are determining the fields resulting from a **spherically symmetric** charge density, we will find that using the **spherical** coordinate system will make our analysis the easiest and most straightforward.

The Position Vector

Consider a point whose location in space is specified with Cartesian coordinates (e.g., P(x,y,z)). Now consider the **directed distance** (a vector quantity!) extending from the origin to this point.



This **particular** directed distance—a vector beginning at the **origin** and extending outward to a point—is a **very important** and fundamental directed distance known as the **position vector** $\bar{r}$.

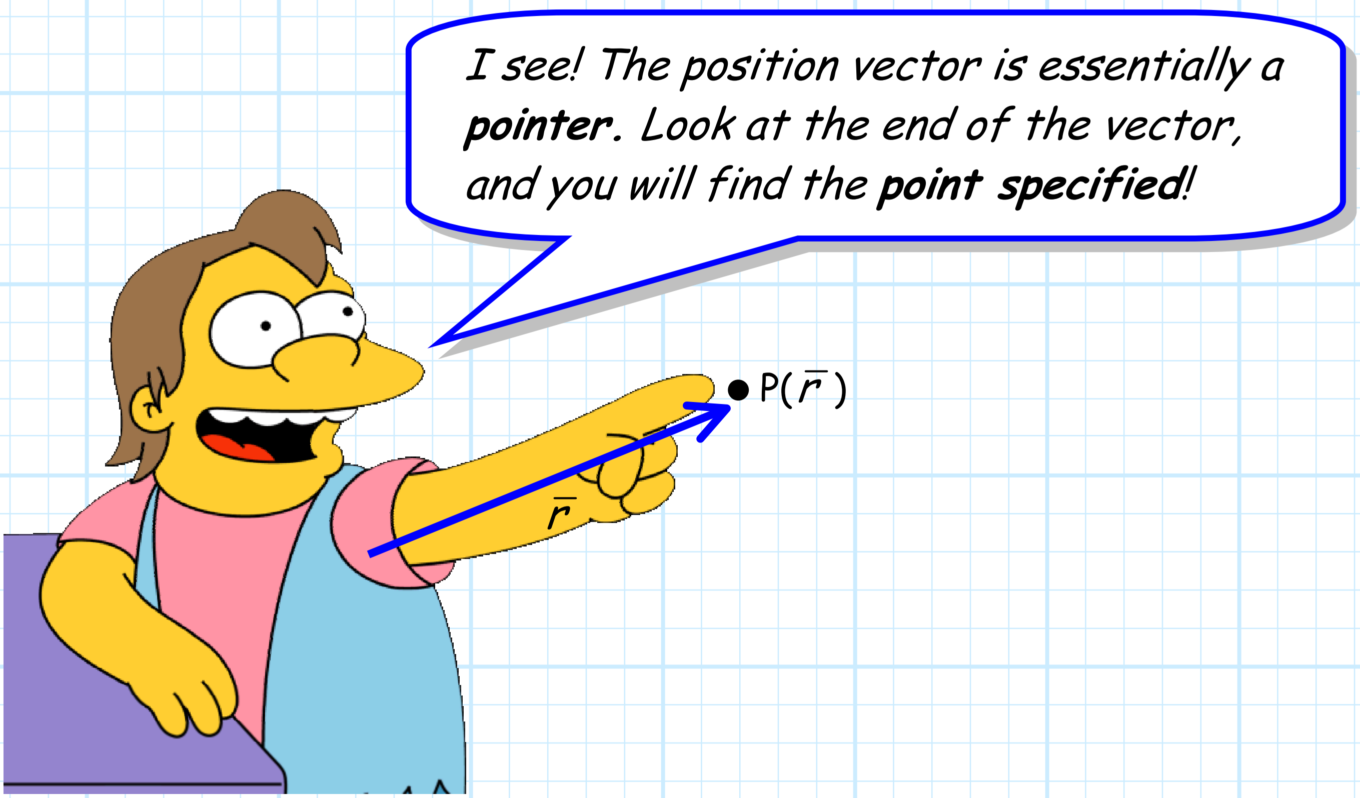
Using the **Cartesian** coordinate system, the position vector can be explicitly written as:

$$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$

* Note that given the **coordinates** of some point (e.g., x=1, y=2, z=-3), we can easily determine the corresponding **position vector** (e.g., $\bar{r} = \hat{a}_x + 2\,\hat{a}_y - 3\,\hat{a}_z$).

* Moreover, given some specific position vector (e.g., $\bar{r} = 4\,\hat{a}_y - 2\,\hat{a}_z$), we can easily determine the corresponding coordinates of that point (e.g., x=0, y=4, z=-2).

In other words, a position vector $\bar{r}$ is an alternative way to denote the location of a point in space! We can use **three coordinate values** to specify a point's location, **or** we can use a **single position vector** $\bar{r}$.



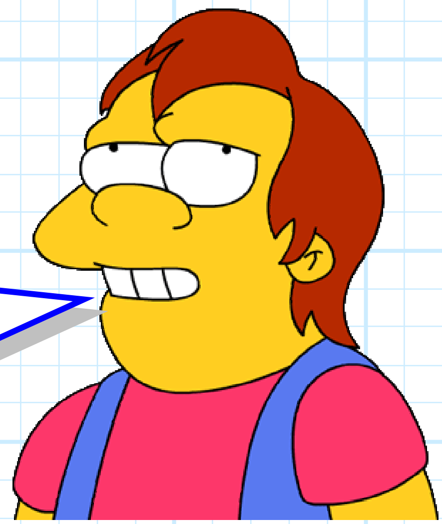
The magnitude of $\bar{r}$

Note the **magnitude** of any and all position vectors is:

$$|\bar{r}| = \sqrt{\bar{r} \cdot \bar{r}} = \sqrt{x^2 + y^2 + z^2} = r$$

The magnitude of the position vector is equal to the **coordinate value** r of the point the position vector is pointing to!

Q: *Hey, this makes **perfect sense**! Doesn't the coordinate value r have a **physical** interpretation as the **distance** between the **point** and the **origin**?*



A: That's right! The **magnitude** of a **directed distance** vector is equal to the **distance** between the two points—in this case the distance between the **specified point** and the **origin**!

Alternative forms of the position vector

Be **careful**! Although the position vector **is correctly** expressed as:

$$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$

It is **NOT CORRECT** to express the position vector as:

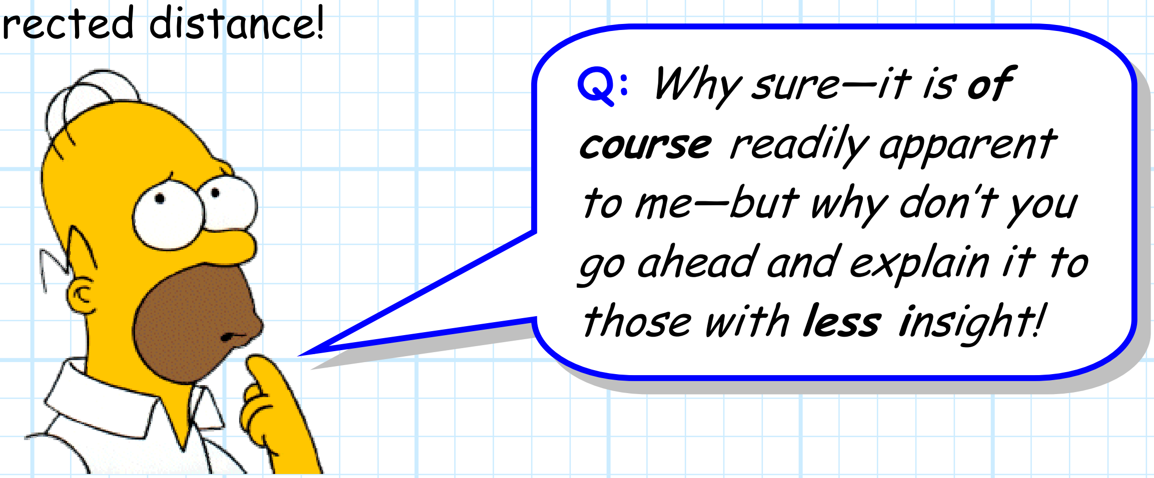
$$\bar{r} \neq \rho\,\hat{a}_p + \phi\,\hat{a}_\phi + z\,\hat{a}_z$$

nor

$$\bar{r} \neq r\,\hat{a}_r + \theta\,\hat{a}_\theta + \phi\,\hat{a}_\phi$$

NEVER, EVER express the position vector in either of these two ways!

It should be **readily apparent** that the two expression above **cannot** represent a position vector—because **neither** is even a directed distance!



A: Recall that the **magnitude** of the position vector $\bar{r}$ has units of **distance**. Thus, the **scalar components** of the position vector must **also** have units of distance (e.g., meters). The coordinates x, y, z, ρ and r **do** have units of distance, but coordinates θ and ϕ do **not**.

Thus, the vectors $\theta\,\hat{a}_\theta$ and $\phi\,\hat{a}_\phi$ **cannot** be vector components of a position vector—or for that matter, any other **directed distance**!

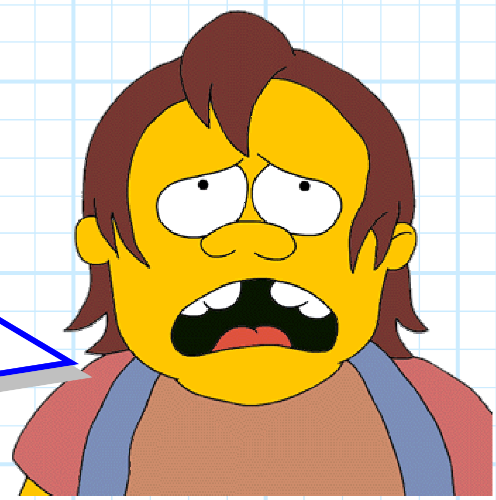
Instead, we can use **coordinate transforms** to show that:

$$\begin{aligned}\bar{r} &= x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\ &= \rho\cos\phi\,\hat{\mathbf{a}}_x + \rho\sin\phi\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\ &= r\sin\theta\cos\phi\,\hat{\mathbf{a}}_x + r\sin\theta\sin\phi\,\hat{\mathbf{a}}_y + r\cos\theta\,\hat{\mathbf{a}}_z\end{aligned}$$

ALWAYS use one of these three expressions of a position vector!!

Note that in **each** of the three expressions above, we use **Cartesian base vectors**. The **scalar components** can be expressed using Cartesian, cylindrical, or spherical **coordinates**, but we must always use **Cartesian base vectors**.

Q: *Why must we **always** use Cartesian base vectors? You said that we could express **any** vector using spherical or base vectors. Doesn't this **also** apply to position vectors?*



A: The reason we **only** use Cartesian base vectors for constructing a position vector is that Cartesian base vectors are the only base vectors whose directions are **fixed**—independent of position in space!

To see why this is important, let's go ahead and **change** the **base vectors** used to express the position vector from Cartesian to spherical or cylindrical. If we do this, we find:

$$\begin{aligned}\bar{r} &= x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\ &= \rho\,\hat{\mathbf{a}}_\rho + z\,\hat{\mathbf{a}}_z \\ &= r\,\hat{\mathbf{a}}_r\end{aligned}$$

Thus, the position vector expressed with the cylindrical coordinate **system** is $\bar{r} = \rho\,\hat{\mathbf{a}}_\rho + z\,\hat{\mathbf{a}}_z$, while with the spherical coordinate **system** we get $\bar{r} = r\,\hat{\mathbf{a}}_r$.

The **problem** with these two expressions is that the direction of base vectors $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_r$ are **not constant**. Instead, they themselves are vector fields—their direction is a function of position!

Thus, an expression such as $\bar{r} = 6\,\hat{\mathbf{a}}_r$ does not explicitly define a point in space, as we do not know in what **direction** base vector $\hat{\mathbf{a}}_r$ is pointing! The expression $\bar{r} = 6\,\hat{\mathbf{a}}_r$ does tell us that the coordinate r=6, but how do we determine what the values of coordinates θ or ϕ are? (*answer: we can't!*)

Compare this to the expression:

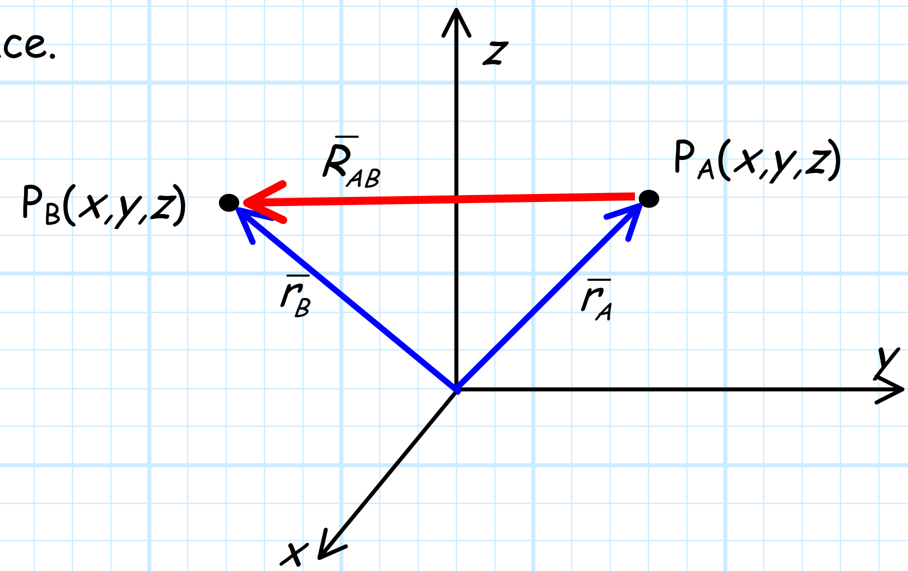
$$\bar{r} \;=\; \hat{a}_x + 2\,\hat{a}_y - 3\,\hat{a}_z$$

Here, the point described by the position vector is **clear** and unambiguous. This position vector identifies the point $P(x=1, y=2, z=-3)$.

Lesson learned: **Always** express a position vector using **Cartesian base vectors** (see box on previous page)!

Applications of the Position Vector

Position vectors are **particularly useful** when we need to determine the directed distance between **two** arbitrary points in space.

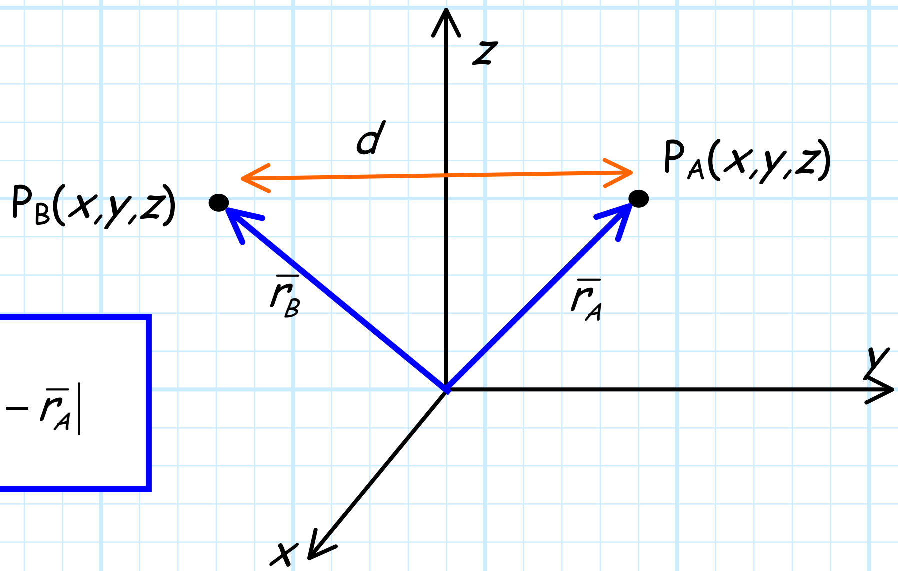


If the location of **point P_A** is denoted by position vector $\bar{r}_A$, and the location of **point P_B** by position vector $\bar{r}_B$, then the **directed distance** from point P_A to point P_B, is:

$$\bar{R}_{AB} = \bar{r}_B - \bar{r}_A$$

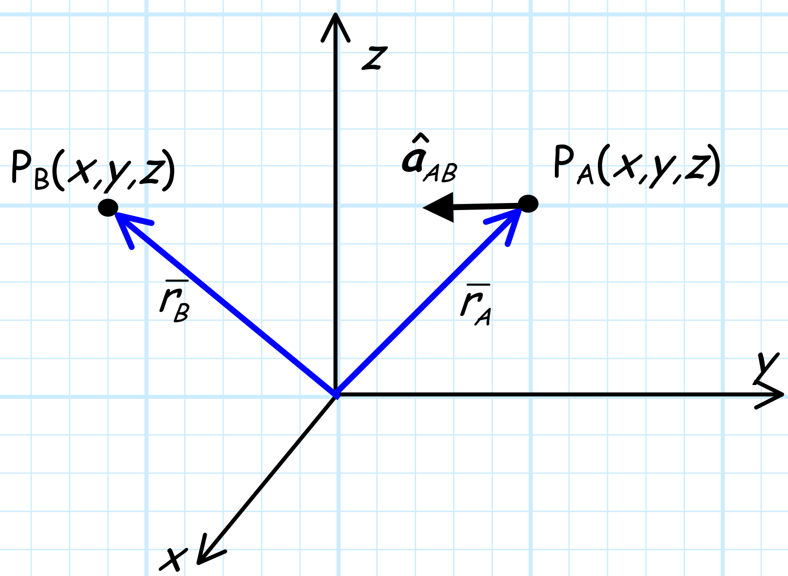
We can use this directed distance $\bar{R}_{AB}$ to describe **much** about the relative locations of point P_A and P_B!

For example, the physical **distance** between these two points is simply the magnitude of this directed distance:



$$d = \left|\bar{R}_{AB}\right| = \left|\bar{r}_B - \bar{r}_A\right|$$

Likewise, we can specify the **direction** toward point P_B, with **respect** to point P_A, by find the **unit vector** $\hat{a}_{AB}$:



$$\hat{a}_{AB} = \frac{\bar{R}_{AB}}{\left|\bar{R}_{AB}\right|} = \frac{\bar{r}_B - \bar{r}_A}{\left|\bar{r}_B - \bar{r}_A\right|}$$

Vector Field Notation

A vector **field** describes a vector value at **every** location in space. Therefore, we can denote a vector field as $\mathbf{A}(x,y,z)$, or $\mathbf{A}(\rho,\phi,z)$, or $\mathbf{A}(r,\theta,\phi)$, explicitly showing that vector quantity **A** is a **function** of position, as denoted by some set of coordinates.

However, as we have emphasized before, the **physical reality** that vector field **A** expresses is independent of the coordinates we use to express it. In other words, although the **math** may look **very different**, we find that:

$$\mathbf{A}(x,y,z) = \mathbf{A}(\rho,\phi,z) = \mathbf{A}(r,\theta,\phi).$$

Alternatively then, we typically express a vector field as simply:

$$\boxed{\mathbf{A}(\bar{r})}$$

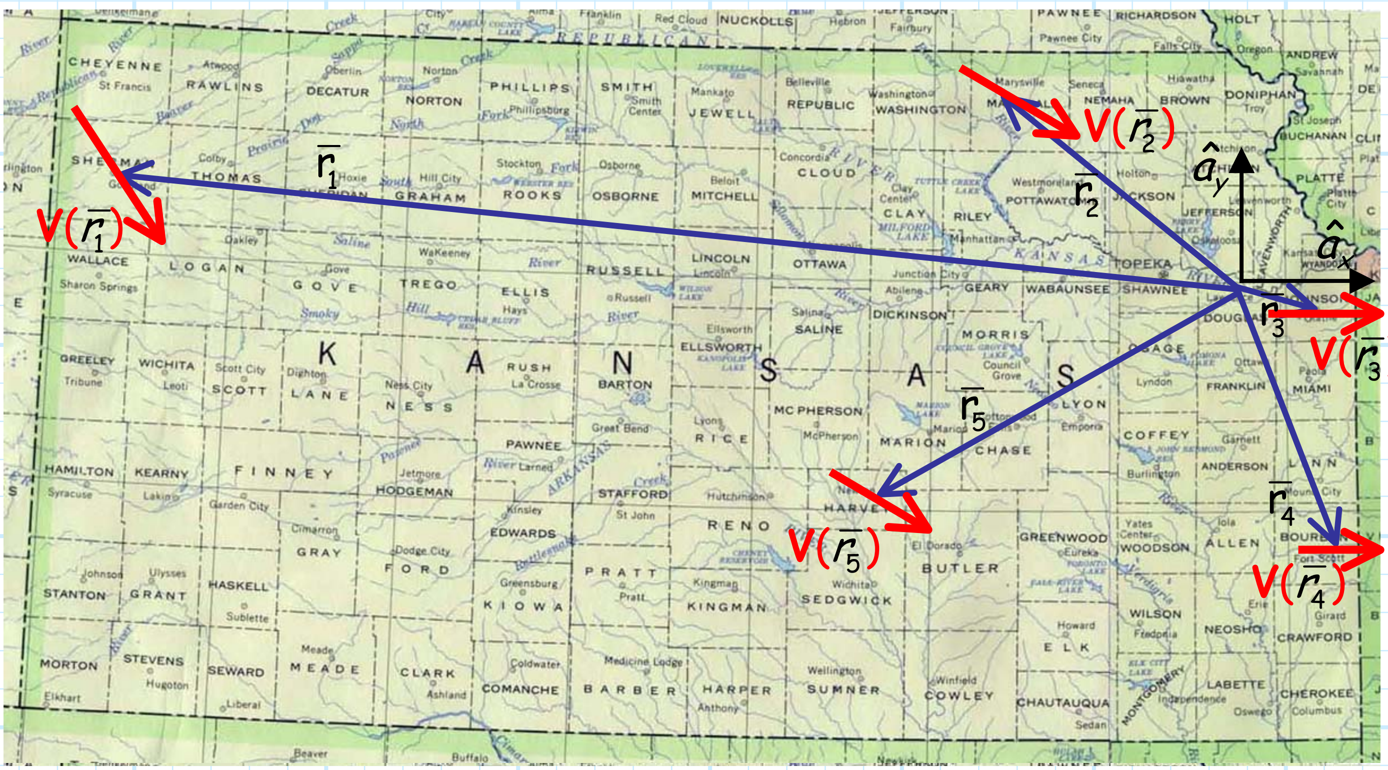
This **symbolically** says everything that we need to convey; vector **A** is a **function** of position—it is a **vector field**!

Note that the vector field notation $\mathbf{A}(\bar{r})$ does **not** explicitly specify a **coordinate system** for expressing **A**. That's up to **you** to decide!

Now, in the vector field expression $\mathbf{A}(\bar{r})$ we note that there are two vectors: **A** and $\bar{r}$. It is **ridiculously important** that you understand what each of these two vectors represents!

> Position vector $\bar{r}$ denotes the location in space where vector **A** is defined.

For example, consider the vector field $\mathbf{V}(\bar{r})$, which describes the **wind velocity** across the state of Kansas.



In this map, the **origin** has been placed at Lawrence. The **locations** of Kansas towns can thus be identified using **position vectors** (units in miles):

$\bar{r}_1 = -400\,\hat{a}_x + 20\,\hat{a}_y$ → the **location** of Goodland, KS

$\bar{r}_2 = -90\,\hat{a}_x + 70\,\hat{a}_y$ → the **location** of Marysville, KS

$\bar{r}_3 = 30\,\hat{a}_x - 5\,\hat{a}_y$ → the **location** of Fort Scott,KS

$\bar{r}_4 = 40\,\hat{a}_x - 90\,\hat{a}_y$ → the **location** of Fort Scott,KS

$\bar{r}_5 = -130\,\hat{a}_x - 70\,\hat{a}_y$ → the **location** of Newton, KS

Evaluating the vector field $\mathbf{V}(\bar{r})$ at these locations provides the wind velocity **at** each Kansas town (units of mph).

$\mathbf{V}(\bar{r}_1) = 15\,\hat{a}_x - 17\,\hat{a}_y$ → the **wind velocity** in Goodland, KS

$\mathbf{V}(\bar{r}_2) = 15\,\hat{a}_x - 9\,\hat{a}_y$ → the **wind velocity** in Marysville, KS

$\mathbf{V}(\bar{r}_3) = 11\,\hat{a}_x$ → the **wind velocity** in Olathe, KS

$\mathbf{V}(\bar{r}_4) = 7\,\hat{a}_x$ → the **wind velocity** in Fort Scott, KS

$\mathbf{V}(\bar{r}_5) = 9\,\hat{a}_x - 4\,\hat{a}_y$ → the **wind velocity** in Newton, KS

Remember, from vector field $\mathbf{A}(\bar{r})$, we can the magnitude and direction of the discrete vector $\mathbf{A}$ that is **located** at the **point** defined by position vector $\bar{r}$.

This discrete vector $\mathbf{A}$ does **not** "extend" from the origin to the point described by position vector $\bar{r}$. Rather, the discrete vector $\mathbf{A}$ describes a quantity **at that point**, and that point only. The magnitude of vector $\mathbf{A}$ does **not** have units of distance! The **length** of the arrow that represents vector $\mathbf{A}$ is merely symbolic—its length has **no** direct physical meaning.

On the other hand, the position vector $\bar{r}$, being a directed distance, **does** extend from the origin to a specific **point** in space. The magnitude of a position vector $\bar{r}$ **is** distance—the length of the **position vector** arrow **has** a direct physical meaning!

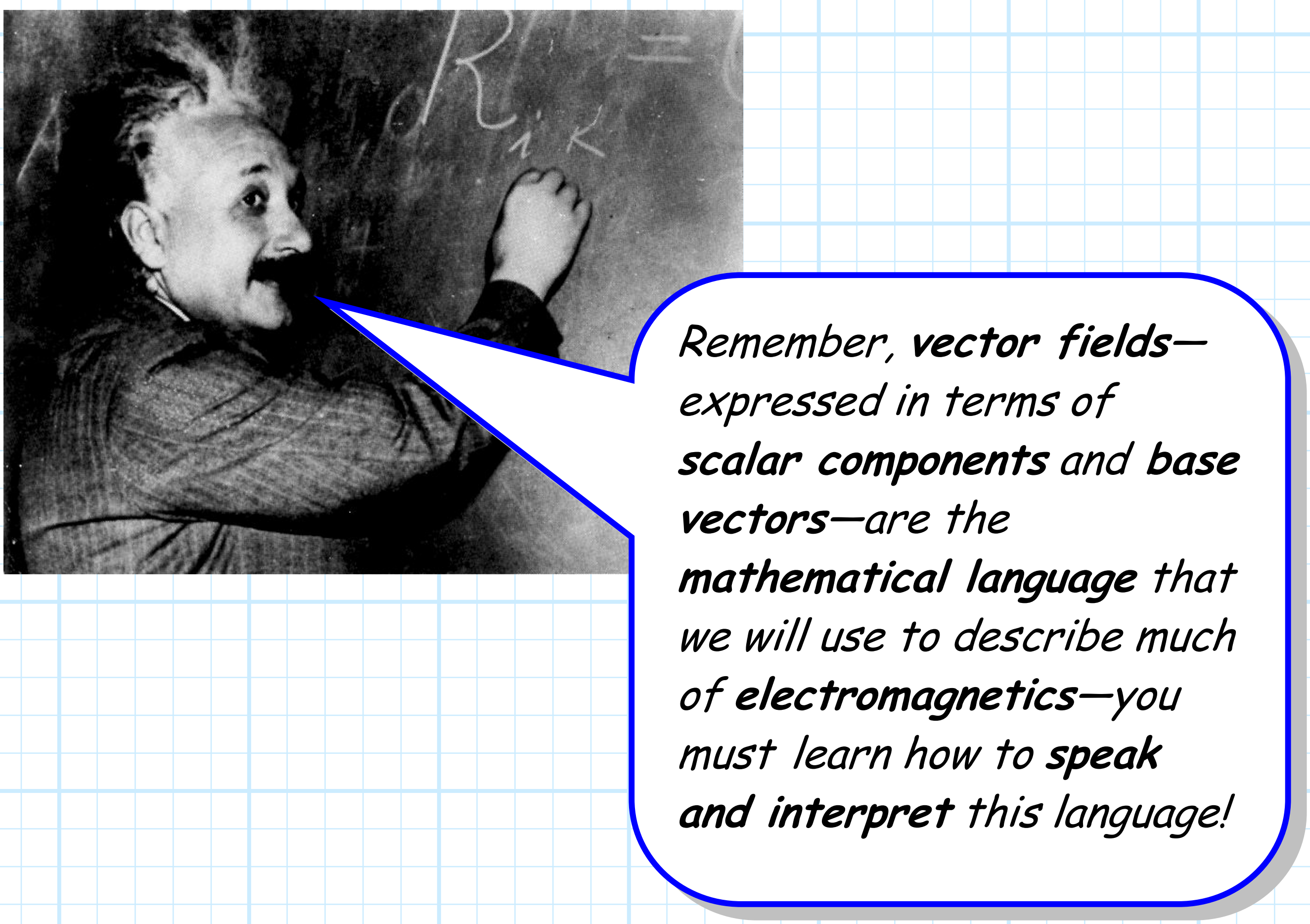
Additionally, we should again note that a vector field need not be static. A **dynamic** vector field is likewise a function of **time**, and thus can be described with the notation:

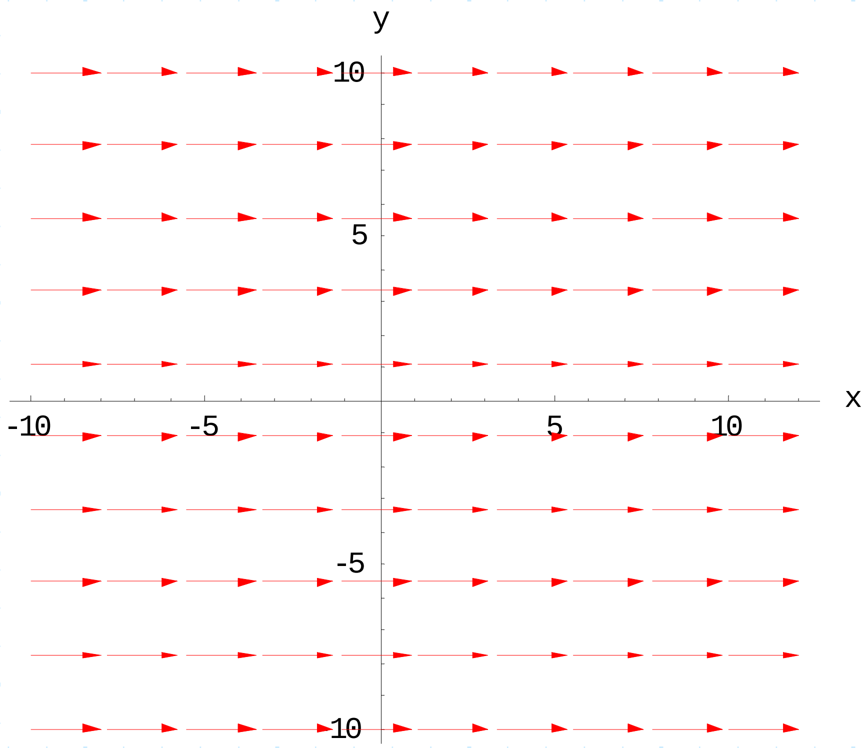
$$\mathbf{A}(\bar{r},t)$$

A Gallery of Vector Fields

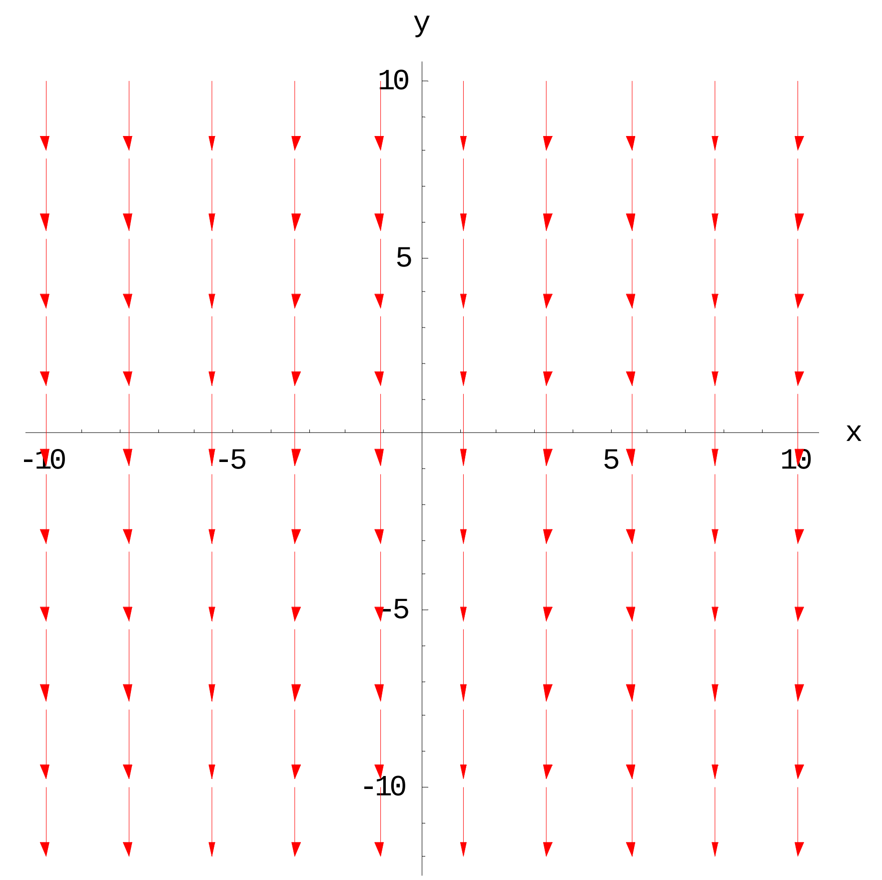
To help **understand** how a vector field relates to its mathematical representation using base vectors, carefully examine and consider these **examples**, plotted on either the ***x-y* plane** (i.e, the plane with all points whose coordinate z=0) or the ***x-z* plane** (i.e, the plane with all points whose coordinate y=0).

Spend some **time** studying each of these examples, until **you** see how the **math** relates to the vector field **plot** and vice versa.

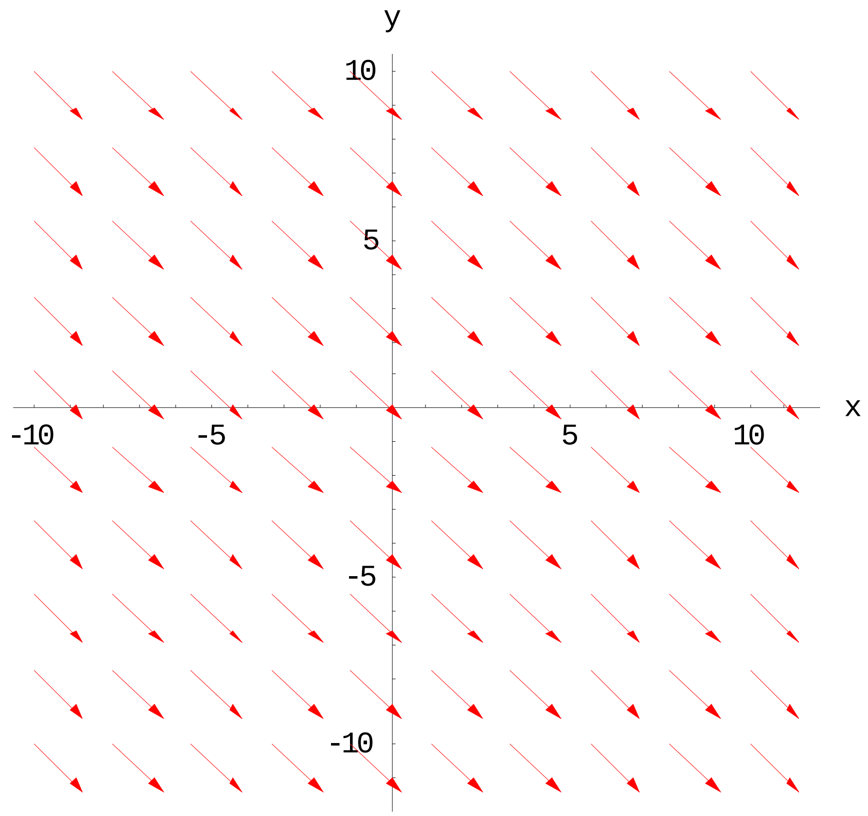




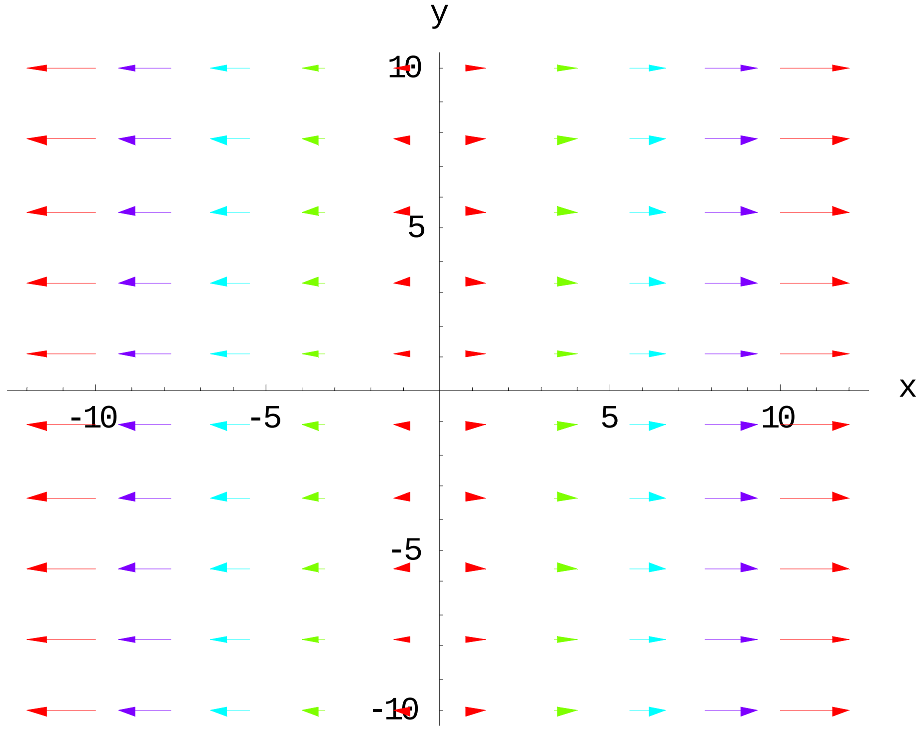
$$\mathbf{A}(\bar{r}) = \hat{\mathbf{a}}_x$$



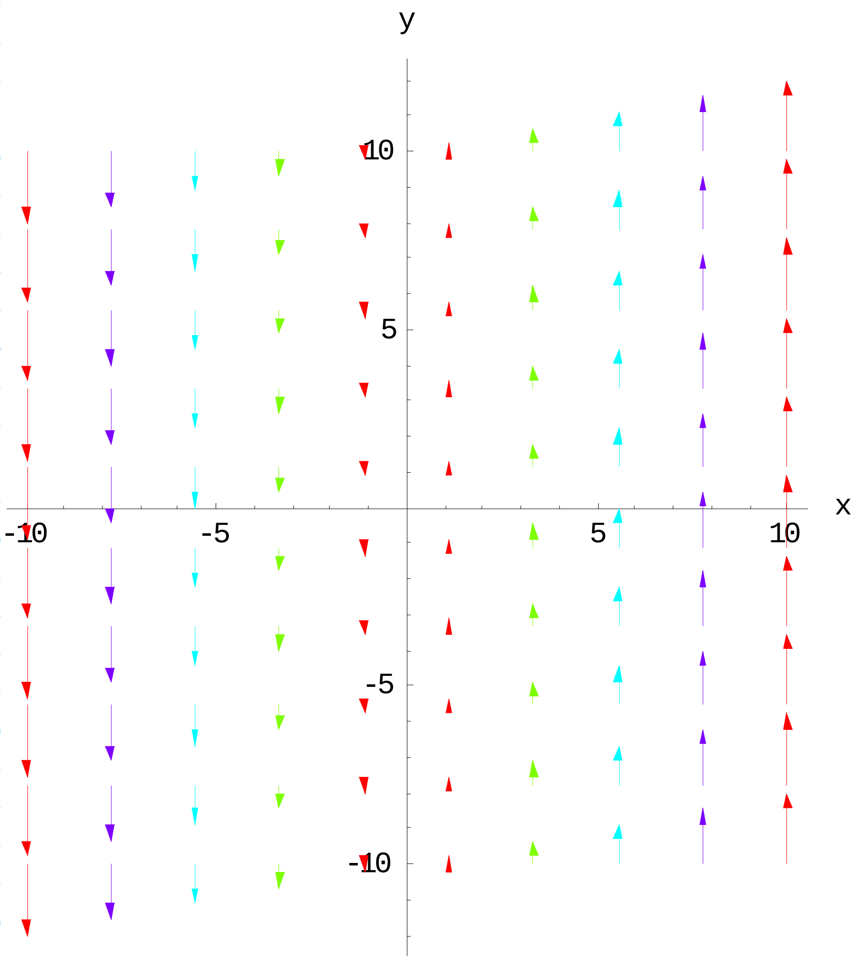
$$\mathbf{A}(\bar{r}) = -\hat{\mathbf{a}}_y$$



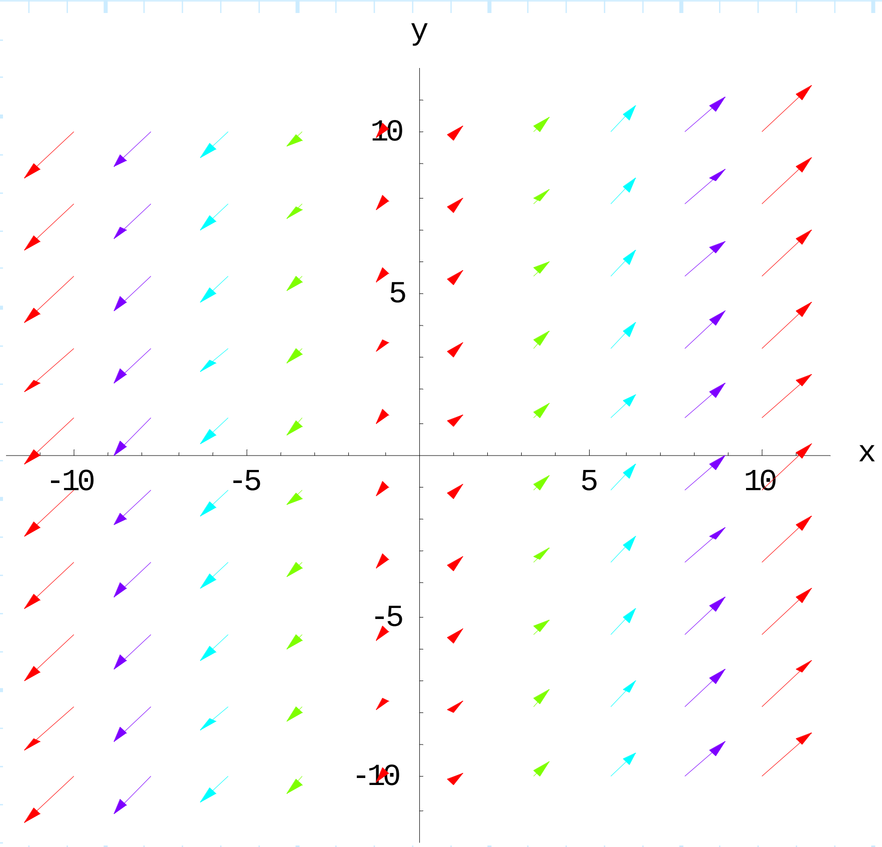
$$\mathbf{A}(\bar{r}) = \hat{a}_x - \hat{a}_y$$



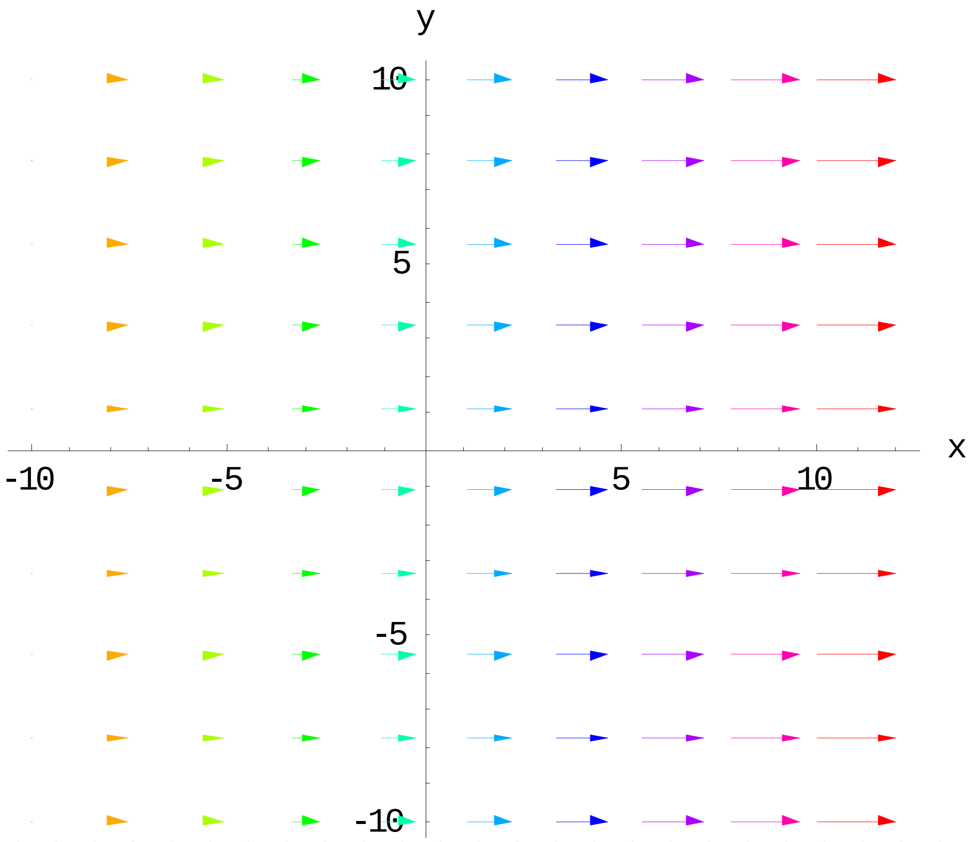
$$\mathbf{A}(\bar{r}) = x\,\hat{a}_x$$



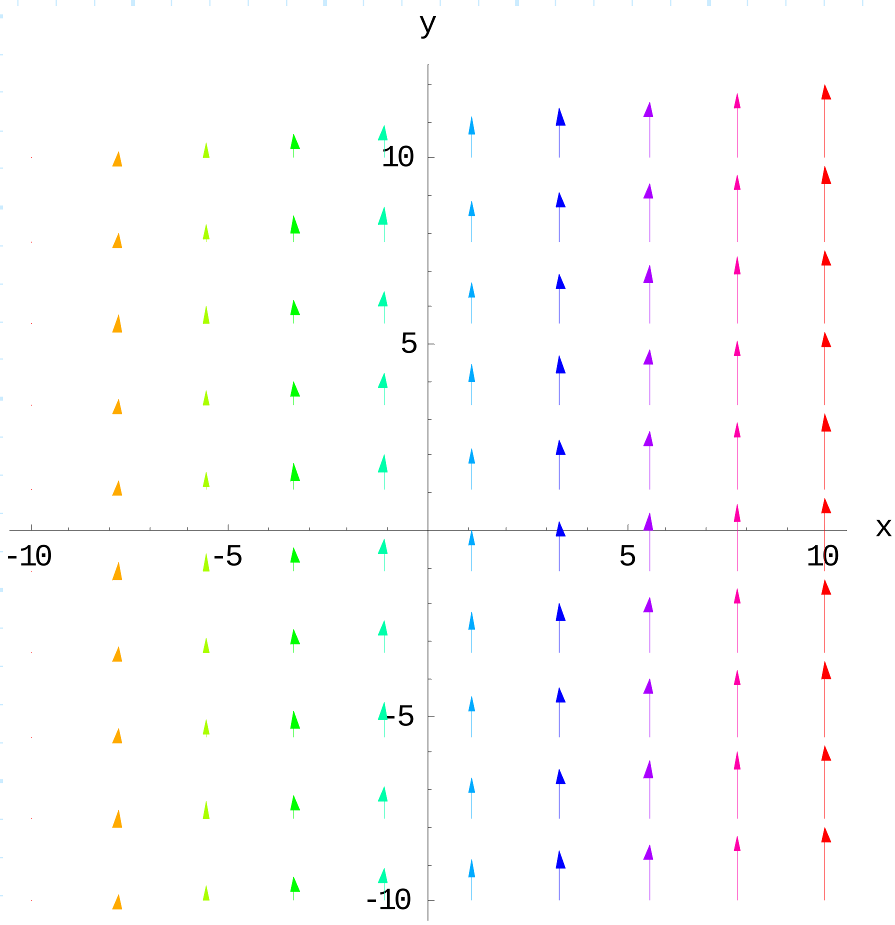
$$\mathbf{A}(\bar{r}) = x\,\hat{a}_y$$



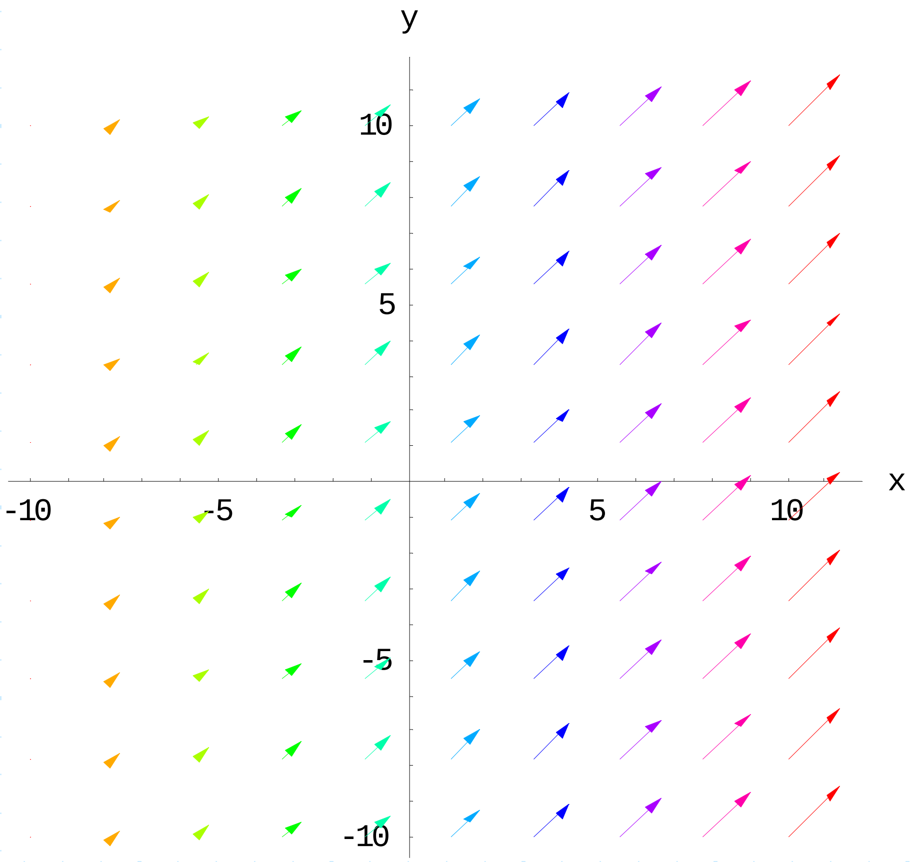
$$\mathbf{A}(\bar{r}) = x\,\hat{a}_x + x\,\hat{a}_y$$



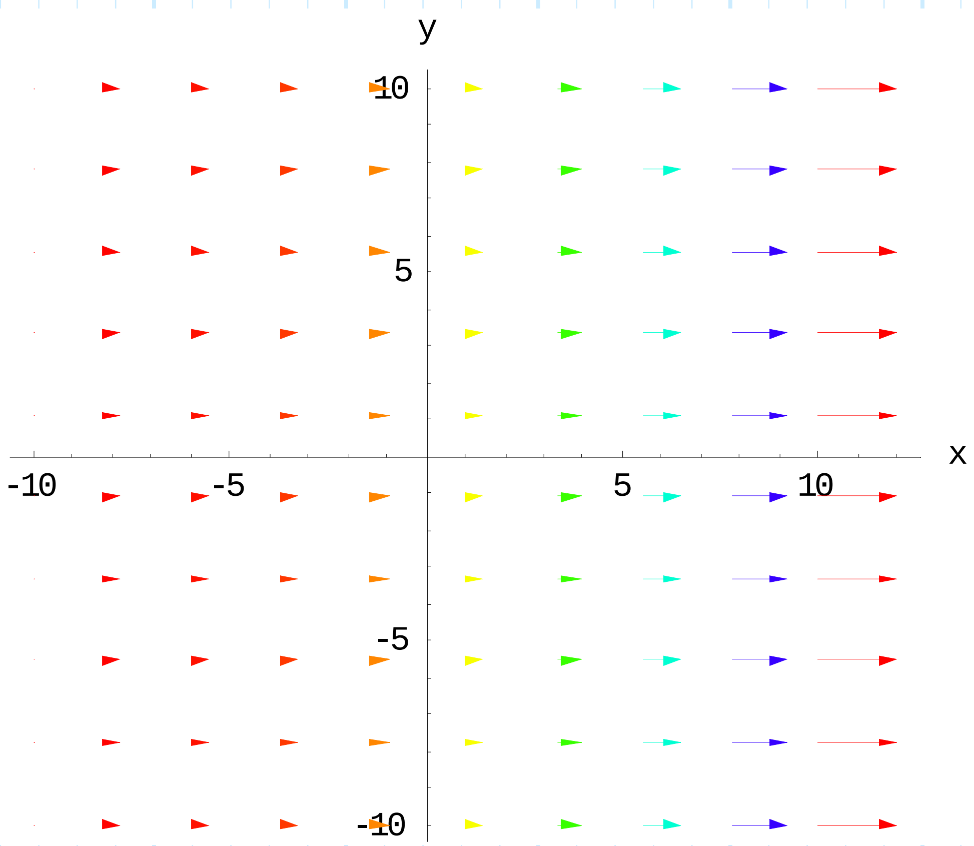
$$\mathbf{A}(\bar{r}) = (10 + x)\hat{\mathbf{a}}_x$$



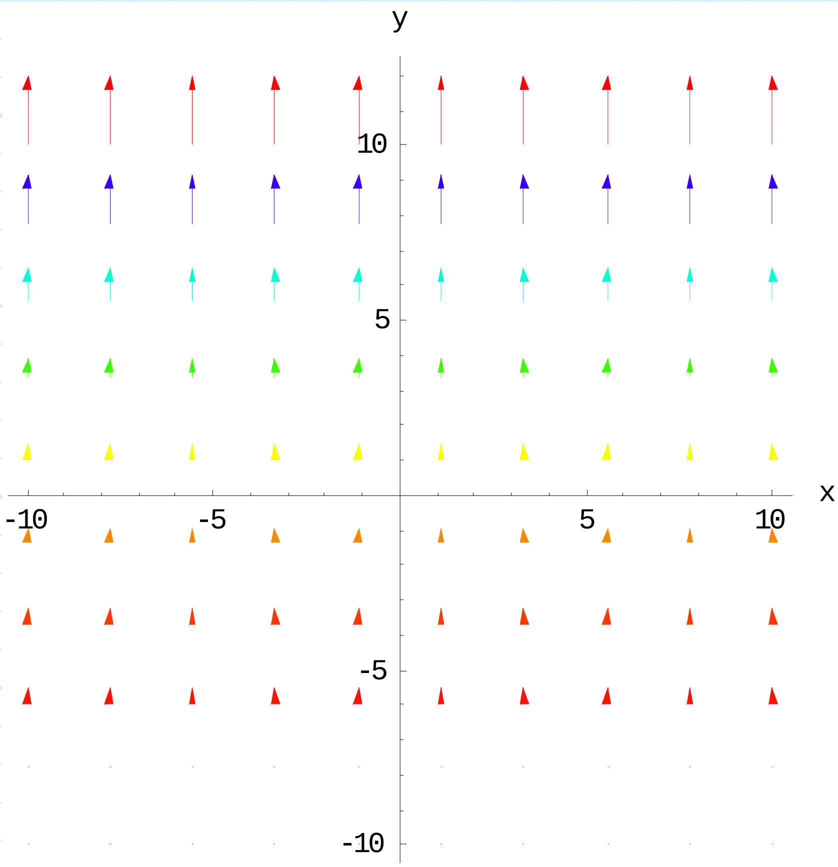
$$\mathbf{A}(\bar{r}) = (10 + x)\hat{\mathbf{a}}_y$$



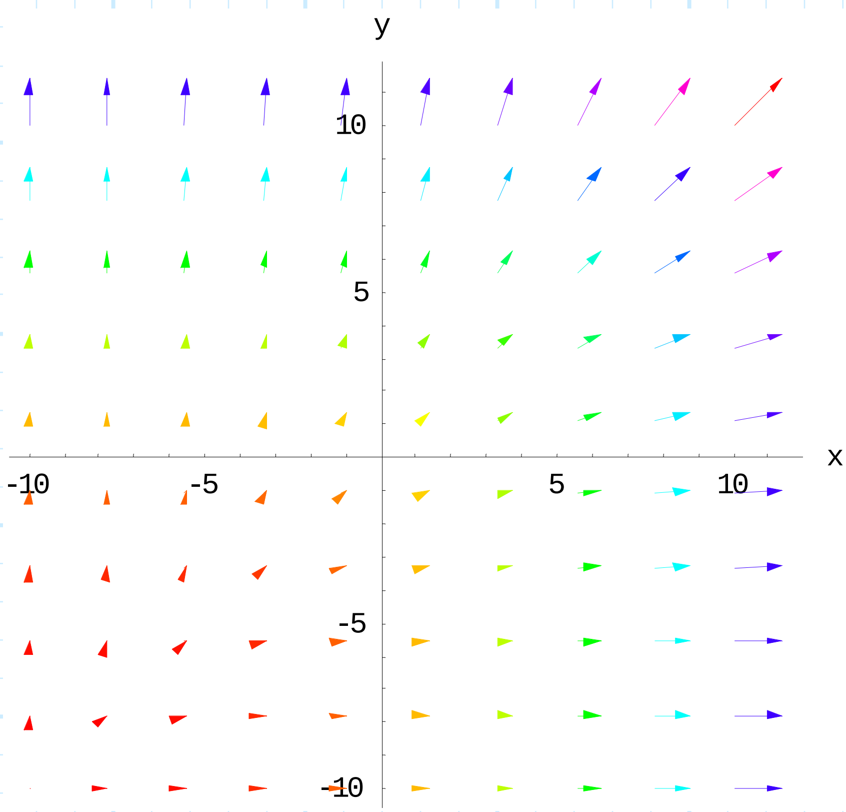
$$\mathbf{A}(\bar{r}) = (10 + x)\hat{\mathbf{a}}_x + (10 + x)\hat{\mathbf{a}}_y$$



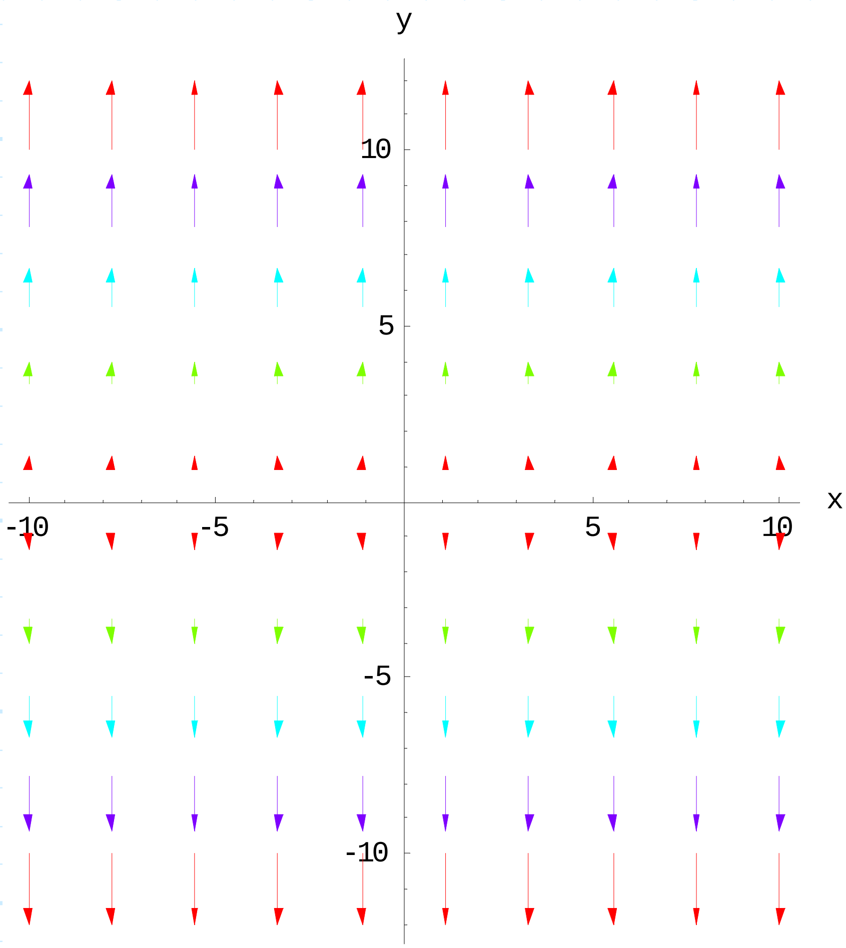
$$\mathbf{A}(\bar{r}) = (10 + x)^3\ \hat{\mathbf{a}}_x$$



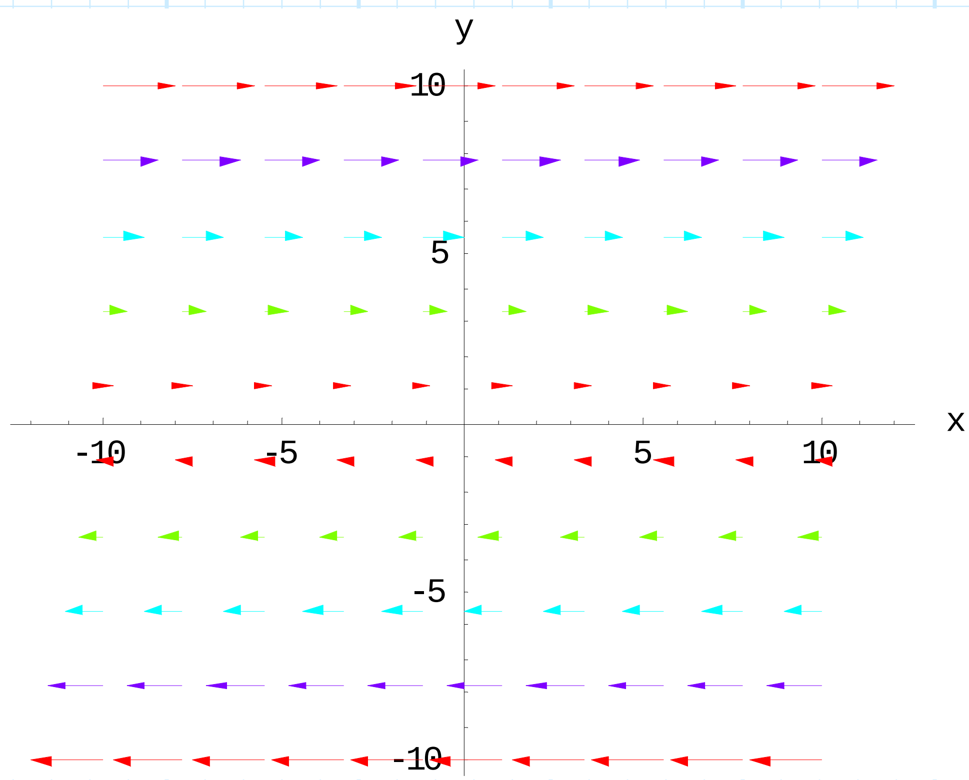
$$\mathbf{A}(\bar{r}) = (10+y)^3 \ \hat{\mathbf{a}}_y$$



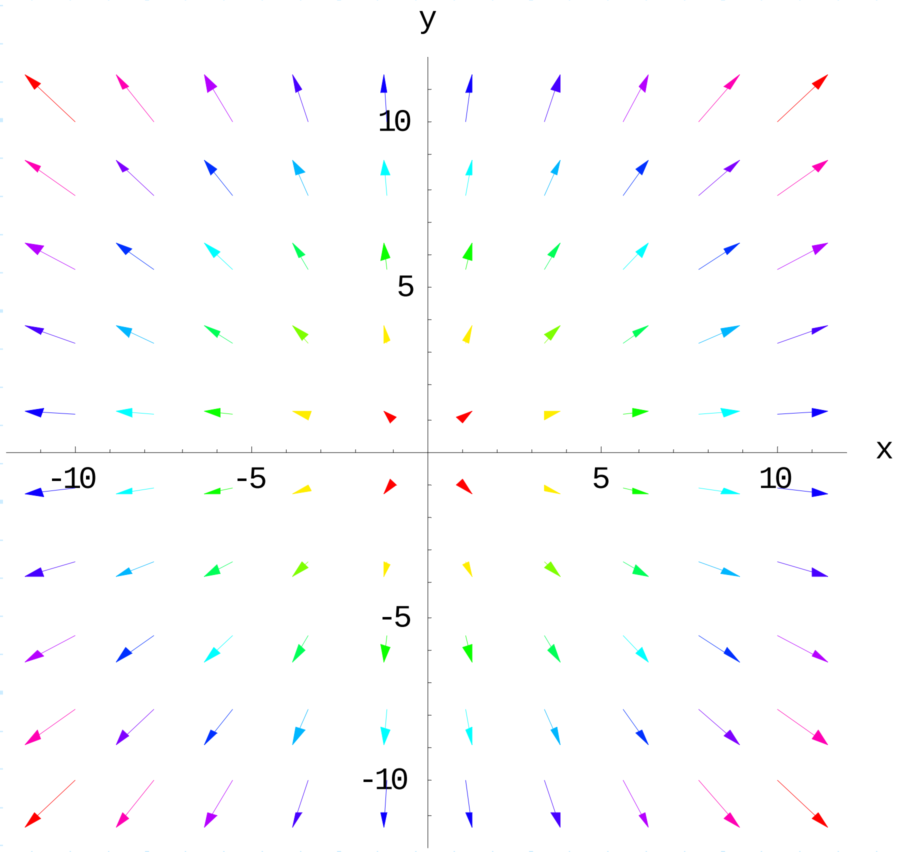
$$\mathbf{A}(\bar{r}) = (10+x)^3 \ \hat{\mathbf{a}}_x + (10+y)^3 \ \hat{\mathbf{a}}_y$$



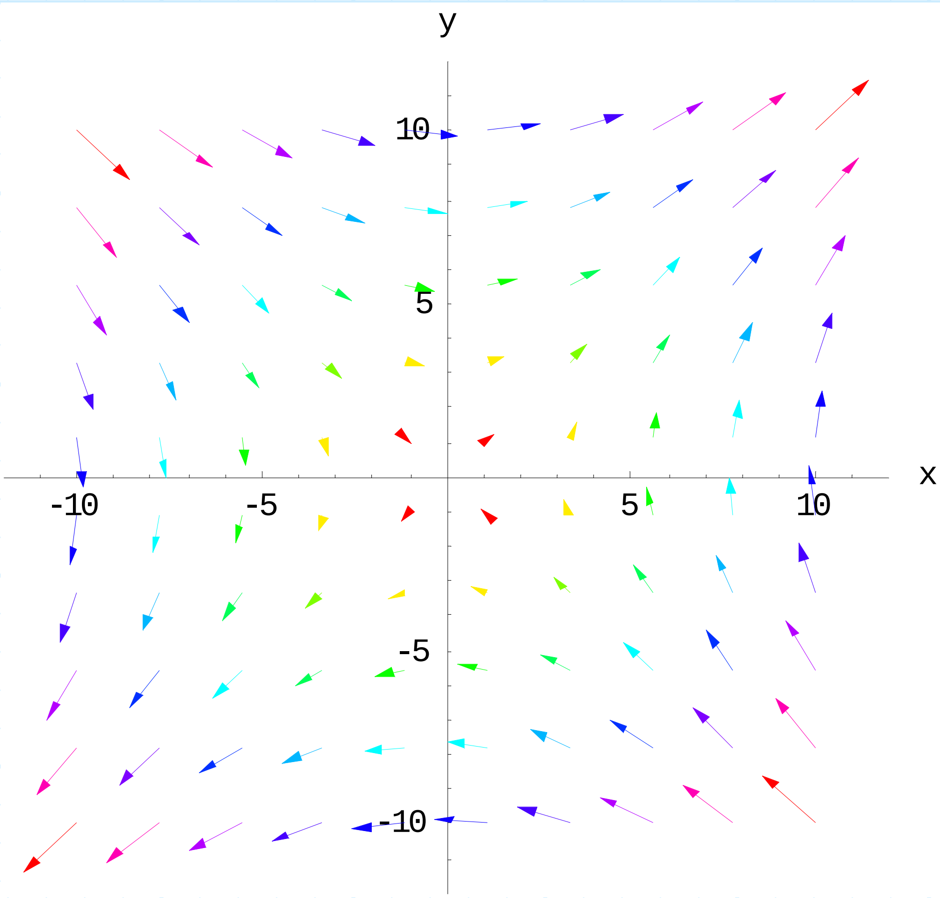
$$\mathbf{A}(\bar{r}) = y\,\hat{\mathbf{a}}_y$$



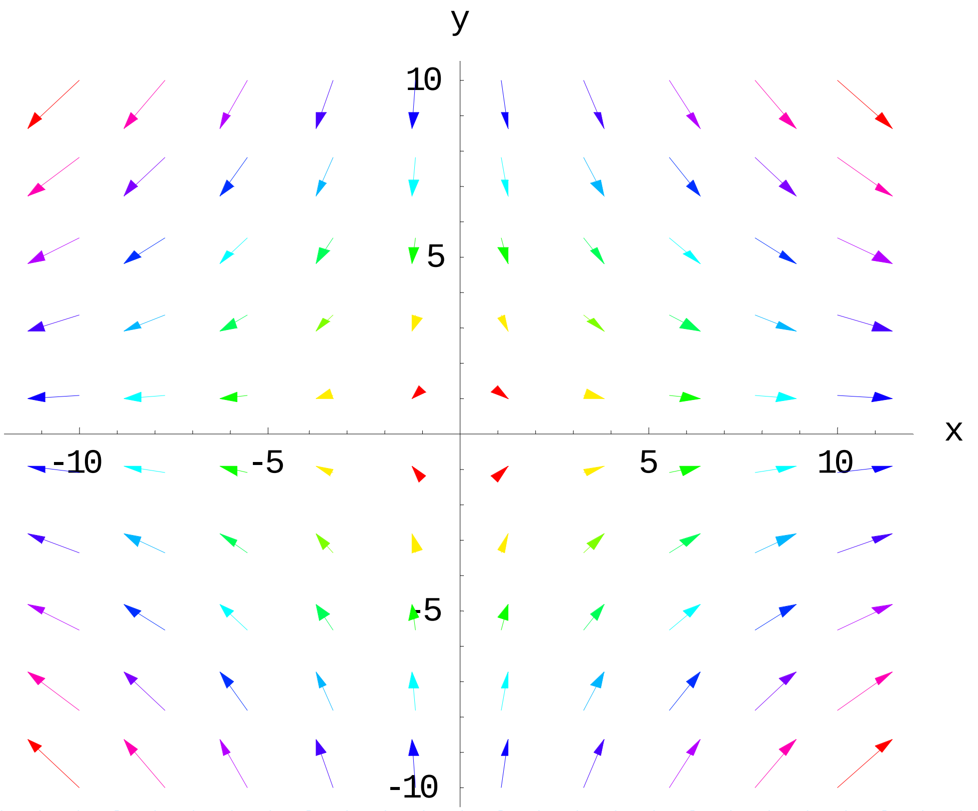
$$\mathbf{A}(\bar{r}) = y\,\hat{\mathbf{a}}_x$$



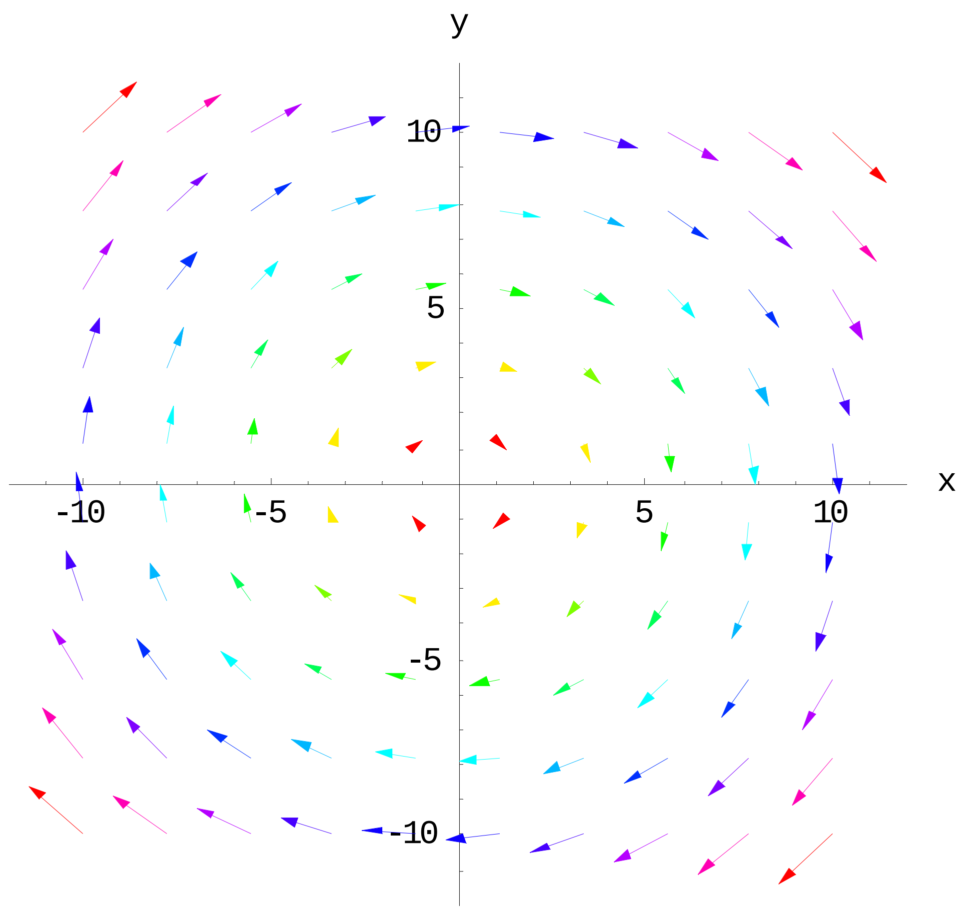
$$\mathbf{A}(\bar{r}) = x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y$$



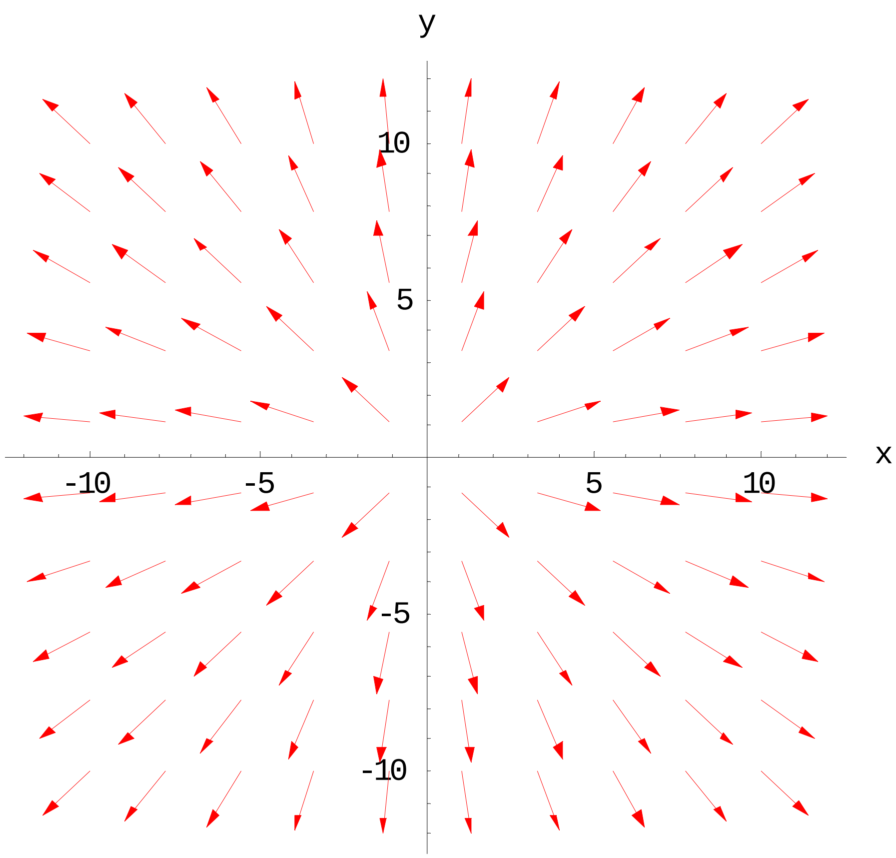
$$\mathbf{A}(\bar{r}) = y\,\hat{\mathbf{a}}_x + x\,\hat{\mathbf{a}}_y$$



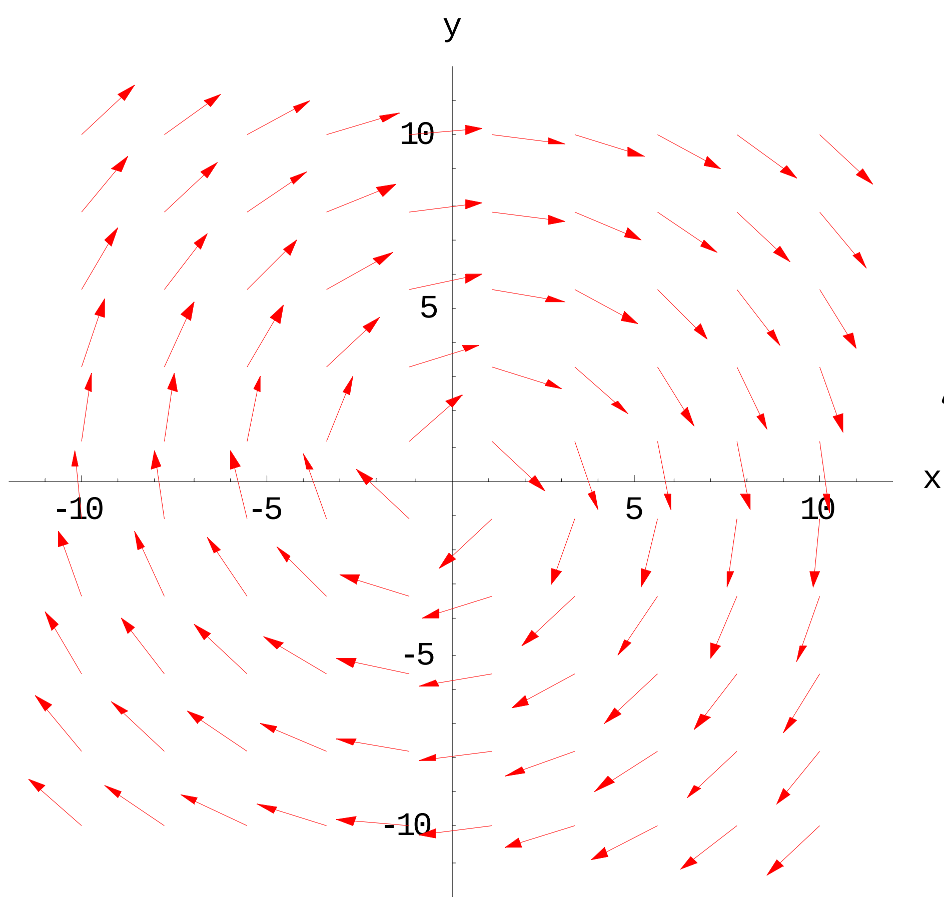
$$\mathbf{A}(\bar{r}) = x\,\hat{a}_x - y\,\hat{a}_y$$



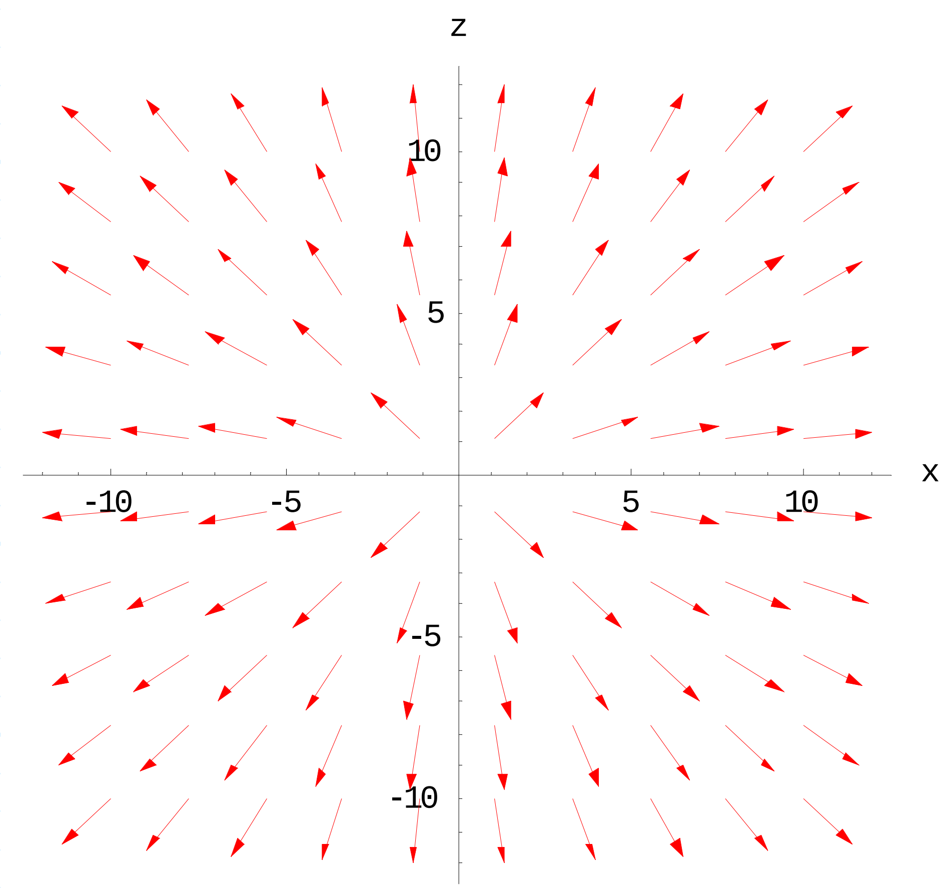
$$\mathbf{A}(\bar{r}) = y\,\hat{a}_x - x\,\hat{a}_y$$



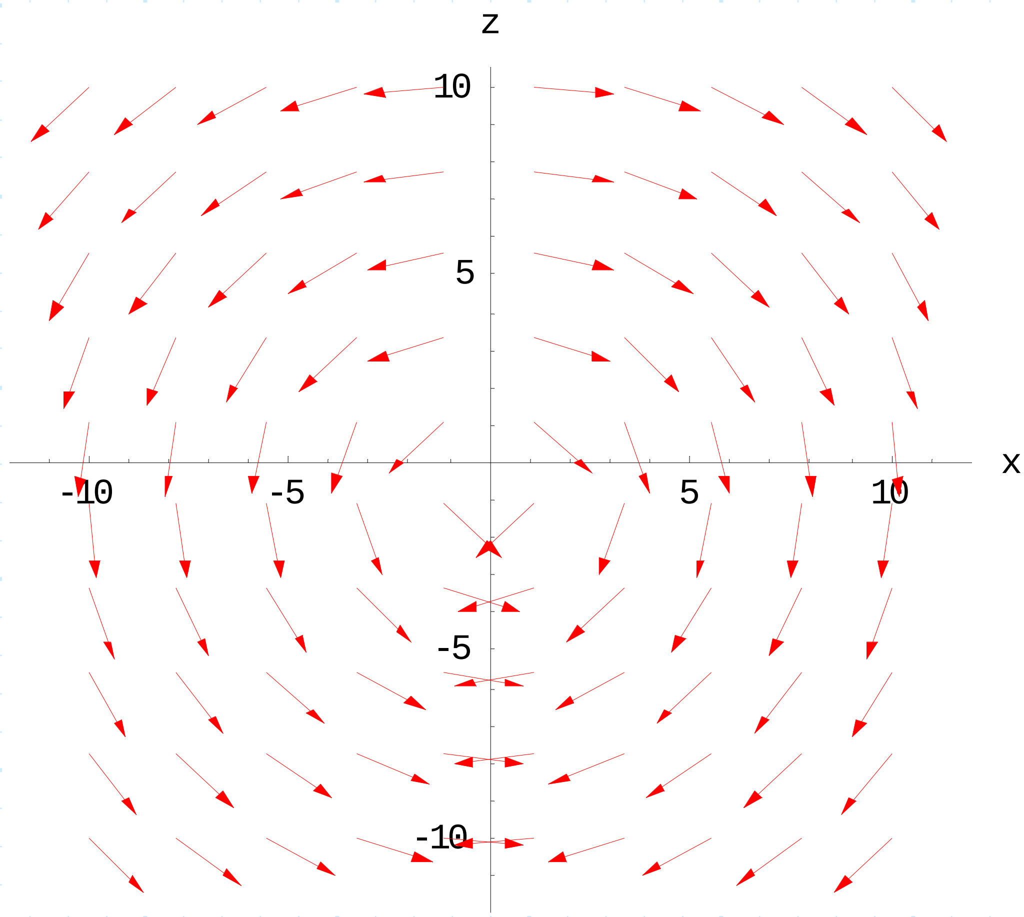
$$\mathbf{A}(\bar{r}) = \hat{a}_\rho$$
$$= \cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y$$



$$\mathbf{A}(\bar{r}) = \hat{a}_\phi$$
$$= \sin\phi\, \hat{a}_x - \cos\phi\, \hat{a}_y$$



$$\mathbf{A}(\bar{\mathbf{r}}) = \hat{\mathbf{a}}_r$$
$$= \sin\theta\cos\phi\,\hat{\mathbf{a}}_x$$
$$+\cos\theta\,\hat{\mathbf{a}}_y$$



$$\mathbf{A}(\bar{\mathbf{r}}) = \hat{\mathbf{a}}_\theta$$
$$= \cos\theta\cos\phi\,\hat{\mathbf{a}}_x$$
$$-\sin\theta\,\hat{\mathbf{a}}_z$$